

## ON EXOTIC AFFINE 3-SPHERES

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ABSTRACT. Every  $\mathbb{A}^1$ -bundle over  $\mathbb{A}_*^2$ , the complex affine plane punctured at the origin, is trivial in the differentiable category but there are infinitely many distinct isomorphism classes of algebraic bundles. Isomorphism types of total spaces of such algebraic bundles are considered; in particular, the complex affine 3-sphere  $\mathbb{S}_{\mathbb{C}}^3$ , given by  $z_1^2 + z_2^2 + z_3^2 + z_4^2 = 1$ , admits such a structure with an additional homogeneity property. Total spaces of nontrivial homogeneous  $\mathbb{A}^1$ -bundles over  $\mathbb{A}_*^2$  are classified up to  $\mathbb{G}_m$ -equivariant algebraic isomorphism and a criterion for nonisomorphism is given. In fact  $\mathbb{S}_{\mathbb{C}}^3$  is not isomorphic as an abstract variety to the total space of any  $\mathbb{A}^1$ -bundle over  $\mathbb{A}_*^2$  of different homogeneous degree, which gives rise to the existence of exotic spheres, a phenomenon that first arises in dimension three. As a by product, an example is given of two biholomorphic but not algebraically isomorphic threefolds, both with a trivial Makar-Limanov invariant, and with isomorphic cylinders.

## INTRODUCTION

Exotic affine spaces emerged in the 1990's as rather unusual objects in affine algebraic geometry. These are smooth complex affine varieties diffeomorphic to a euclidean space but not algebraically isomorphic to the usual affine space. Actually, the first examples were constructed by Ramanujam in a landmark paper [24] in which he also established the non existence of exotic affine planes. Since then, many other examples of smooth contractible affine varieties of any dimension  $n \geq 3$  have been discovered and these objects have progressively become ubiquitous in affine algebraic geometry. The study of these potential exotic  $\mathbb{A}^n$ 's has been a motivation for the introduction and the development of new techniques and "designer" invariants which in turn led to important progress in related questions, such as the Zariski Cancellation Problem (see e.g. [25] for a survey). So far, these invariants have succeeded in distinguishing certain of these varieties from usual affine spaces, most notably the famous Russell cubic threefold  $X = \{x^2y + z^2 + x + t^3 = 0\} \subset \mathbb{A}^4$  [17, 19]. But the main difficulty still remains the lack of effective tools to recognize exotic spaces or, equivalently, the lack of effective characterizations of affine spaces among affine varieties.

More generally, given any smooth complex affine variety  $V$ , one can ask if there exists smooth affine varieties  $W$  non isomorphic to  $V$  but which are biholomorphic or diffeomorphic to  $V$  when equipped with their underlying structures of complex analytic or differentiable manifolds. When such exist, these varieties  $W$  could be called exotic algebraic structures on  $V$ , but it makes more sense to reserve this terminology for the case where the chosen variety  $V$  carries an algebraic structure that we consider as the "usual" one.

In addition to affine spaces, a very natural class for which we have such "usual" algebraic structures consists of non-degenerate smooth complex affine quadrics, i.e., varieties isomorphic to one of the form  $\mathbb{S}_{\mathbb{C}}^n = \{x_1^2 + \cdots + x_{n+1}^2 = 1\}$  equipped with its unique structure of a closed algebraic subvariety of  $\mathbb{A}_{\mathbb{C}}^{n+1}$ . So an *exotic complex affine  $n$ -sphere*, if it exists, will be a smooth complex affine variety diffeomorphic to  $\mathbb{S}_{\mathbb{C}}^n$  but not algebraically isomorphic to it. Since  $\mathbb{S}_{\mathbb{C}}^1 \simeq \mathbb{A}^1 \setminus \{0\}$  is the unique smooth affine curve  $C$  with  $H_1(C, \mathbb{Z}) \simeq \mathbb{Z}$ , there is no exotic affine 1-sphere. Similarly, there is no exotic affine 2-sphere and the same phenomenon as for affine spaces occurs: the algebraic structure on a smooth affine surface diffeomorphic to  $\mathbb{S}_{\mathbb{C}}^2$  is actually uniquely determined by its topology: namely, a smooth affine surface  $S$  is algebraically isomorphic to  $\mathbb{S}_{\mathbb{C}}^2$  if and only if it has the same homology type and the same homotopy type at infinity as  $\mathbb{S}_{\mathbb{C}}^2$  (see 3.3 in the appendix below).

In the context of the cancellation problem for factorial threefolds, S. Maubach and the second author [7] studied a family of smooth affine threefolds with the homology type of  $\mathbb{S}_{\mathbb{C}}^3$ : starting from a Brieskorn surface  $S_{p,q,r} = \{x^p + y^q + z^r = 0\} \subset \mathbb{A}^3$ , they consider smooth affine threefolds  $Z_{m,n} \subset S_{p,q,r} \times \mathbb{A}^2$  defined by equations of the form  $x^m v - y^n u = 1$ ,  $m \geq n \geq 1$ . These varieties come equipped via the first projection with the structure of a locally trivial  $\mathbb{A}^1$ -bundle  $\rho: Z_{m,n} \rightarrow S_{p,q,r}^*$  over the smooth locus  $S_{p,q,r}^* = S_{p,q,r} \setminus \{(0,0,0)\}$  of  $S_{p,q,r}$ . For a fixed triple  $(p,q,r)$ , they are all diffeomorphic to each other and have the Brieskorn sphere  $\Sigma(p,q,r)$  as a strong deformation retract. The main result of [7] asserts in contrast that if  $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} < 1$ , then the isomorphism type of the total space of an  $\mathbb{A}^1$ -bundle over  $S_{p,q,r}^*$  as an abstract algebraic variety is uniquely determined by its isomorphism class as an  $\mathbb{A}^1$ -bundle over  $S_{p,q,r}^*$ .

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up to composition by automorphisms of  $S_{p,q,r}^*$ .<sup>1</sup> This enables in particular the conclusion that the  $Z_{m,n}$  are pairwise non isomorphic algebraic varieties, despite the isomorphism of all of the cylinders  $Z_{m,n} \times \mathbb{A}^1$ .

Noting that  $\mathbb{S}_{\mathbb{C}}^3 \cong \mathrm{SL}_2(\mathbb{C}) = \{xv - yu = 1\} \subset \mathbb{A}_*^2 \times \mathbb{A}^2$ , where  $\mathbb{A}_*^2 = \mathrm{Spec}(\mathbb{C}[x, y]) \setminus \{(0, 0)\}$ , the previous result strongly suggests that the varieties

$$X_{m,n} = \{x^m v - y^n u = 1\} \subset \mathbb{A}_*^2 \times \mathbb{A}^2, \quad m + n > 2,$$

could be exotic affine 3-spheres. Indeed, for every  $m, n \geq 1$ , the first projection again induces a Zariski locally trivial  $\mathbb{A}^1$ -bundle  $\rho : X_{m,n} \rightarrow \mathbb{A}_*^2$ . The latter being trivial in the euclidean topology, the  $X_{m,n}$  are thus all diffeomorphic to  $\mathbb{A}_*^2 \times \mathbb{R}^2$  and have the real sphere  $S^3$  as a strong deformation retract. This holds more generally for any Zariski locally trivial  $\mathbb{A}^1$ -bundle  $\rho : X \rightarrow \mathbb{A}_*^2$ , and so all smooth affine threefolds admitting such a structure are natural candidates for being exotic 3-spheres. But it turns out that it is a challenging problem to distinguish these varieties from  $\mathbb{S}_{\mathbb{C}}^3 \simeq X_{1,1}$  since, in contrast with the situation considered in [7], the isomorphism type of the total space of an  $\mathbb{A}^1$ -bundle  $\rho : X \rightarrow \mathbb{A}_*^2$  as an abstract variety is no longer uniquely determined by its structure as an  $\mathbb{A}^1$ -bundle; for instance, a consequence of our main result is the following rather unexpected fact that for every pair  $(m, n), (p, q) \in \mathbb{Z}_{>0}^2$  such that  $m + n = p + q \geq 4$ , the threefolds  $X_{m,n}$  and  $X_{p,q}$  are isomorphic as abstract algebraic varieties while they are isomorphic as  $\mathbb{A}^1$ -bundles over  $\mathbb{A}_*^2$  only if  $\{m, n\} = \{p, q\}$ .<sup>2</sup>

While a complete and effective classification of isomorphism types of total spaces of  $\mathbb{A}^1$ -bundles over  $\mathbb{A}_*^2$  seems out of reach for the moment, we obtain a satisfactory answer for a particular class of bundles containing the varieties  $X_{m,n}$  that we call *homogeneous  $\mathbb{G}_a$ -bundles*. These are principal  $\mathbb{G}_a$ -bundles  $\rho : X \rightarrow \mathbb{A}_*^2$  equipped with a lift of the  $\mathbb{G}_m$ -action  $\lambda \cdot (x, y) = (\lambda x, \lambda y)$  on  $\mathbb{A}_*^2$  which is “locally linear” on the fibers of  $\rho$ . This holds for instance on the  $X_{m,n}$ ’s for the lifts  $\lambda \cdot (x, y, u, v) = (\lambda x, \lambda y, \lambda^{-n} u, \lambda^{-m} v)$ , which are the analogues of the action of the maximal torus of  $X_{1,1} = \mathrm{SL}_2(\mathbb{C})$  on  $\mathrm{SL}_2(\mathbb{C})$  by multiplication on the right. For such bundles, there is natural notion of homogeneous degree for which, in particular, the bundle  $X_{m,n} \rightarrow \mathbb{A}_*^2$  equipped with the previous lift has homogeneous degree  $-m - n$ . Our main classification result then reads as follows (see Theorem 2.3):

**Theorem.** *The total spaces of two nontrivial homogeneous  $\mathbb{G}_a$ -bundles are  $\mathbb{G}_m$ -equivariantly isomorphic if and only if they have the same homogeneous degree. In particular, for a fixed  $d \geq 2$ , the total spaces of nontrivial homogeneous  $\mathbb{G}_a$ -bundles  $\rho : X \rightarrow \mathbb{A}_*^2$  of degree  $-d$  are all isomorphic as abstract affine varieties.*

This implies in particular that a variety  $X_{m,n}$  with  $m + n \geq 3$  equipped with the action above is not  $\mathbb{G}_m$ -equivariantly isomorphic to  $X_{1,1}$ . We finally derive from a careful study of the effect of algebraic isomorphisms on the algebraic de Rham cohomology of total spaces of  $\mathbb{A}^1$ -bundles over  $\mathbb{A}_*^2$  that every variety  $X_{m,n}$  with  $m + n \geq 3$  is indeed an exotic affine 3-sphere. The criterion we give in Theorem 2.5 also provides an effective tool to construct families of pairwise non isomorphic exotic 3-spheres : for instance, we show that the varieties  $X_{2,2} = \{x^2 v - y^2 u = 1\}$  and  $\tilde{X}_{2,2} = \{x^2 v - y^2 u = 1 + xy\}$  are non isomorphic exotic affine 3-spheres yet they are even biholomorphic as complex analytic manifolds.

The article is organized as follows. In the first section, the basic properties of  $\mathbb{A}^1$ -bundles over the punctured plane  $\mathbb{A}_*^2$  are reviewed and the notion of homogeneous  $\mathbb{G}_a$ -bundle is developed. The second section is devoted to the proofs of the various isomorphism criteria presented above. The third section takes the form of an appendix, in which we give a short proof of the non existence of exotic affine 2-spheres and establish a refined version of the so-called Danilov-Gizatullin Isomorphism Theorem [10] which is used in the proof of the main Theorem 2.3

## 1. BASIC FACTS ON $\mathbb{A}^1$ -BUNDLES AND $\mathbb{G}_a$ -BUNDLES OVER $\mathbb{A}_*^2$

### 1.1. Recollection on algebraic $\mathbb{A}^1$ -bundles and $\mathbb{G}_a$ -bundles.

An  $\mathbb{A}^1$ -bundle over a scheme  $S$  is a morphism  $\rho : X \rightarrow S$  for which every point of  $S$  has a Zariski open neighborhood  $U \subset S$  with a local trivialization such that  $\rho^{-1}(U) \simeq U \times \mathbb{A}^1$  as schemes over  $U$ . Transition isomorphisms over the intersections  $U_i \cap U_j$  of pairs of such open sets are given by affine transformations of the fiber isomorphism classes of such bundles are thus in one-to-one correspondence with isomorphism classes of Zariski locally trivial principal bundles under the affine group  $\mathrm{Aut}(\mathbb{A}^1) \simeq \mathbb{G}_m \ltimes \mathbb{G}_a$ . Additional properties of these bundles can be read from the exact sequence of non-abelian cohomology

$$0 \rightarrow H^0(S, \mathbb{G}_a) \rightarrow H^0(S, \mathbb{G}_m \ltimes \mathbb{G}_a) \rightarrow H^0(S, \mathbb{G}_m) \rightarrow H^1(S, \mathbb{G}_a) \rightarrow H^1(S, \mathbb{G}_m \ltimes \mathbb{G}_a) \rightarrow H^1(S, \mathbb{G}_m)$$

<sup>1</sup>In loc. cit., this property is established by algebraic methods involving the computation of the Makar-Limanov invariant of the varieties  $Z_{m,n}$ , but this can also be seen alternatively as consequence of the fact for  $1/p + 1/q + 1/r < 1$ , the logarithmic Kodaira dimension  $\bar{\kappa}(S_{p,q,r}^*)$  of  $S_{p,q,r}^*$  is positive.

<sup>2</sup>This fact was actually already observed by the authors and P.D Metha in the unpublished paper [5].

deduced from the short exact sequence of groups  $0 \rightarrow \mathbb{G}_a \rightarrow \mathbb{G}_m \times \mathbb{G}_a \rightarrow \mathbb{G}_m \rightarrow 0$  (see e.g. [9, 3.3.1]). For instance, if  $S$  is affine then  $H^1(S, \mathbb{G}_a) \simeq H^1(S, \mathcal{O}_S) = \{0\}$  and so every  $\mathbb{A}^1$ -bundle over  $S$  actually carries the structure of a line bundle. Similarly, if the Picard group  $\text{Pic}(S) \simeq H^1(S, \mathbb{G}_m)$  of  $S$  is trivial then every  $\mathbb{A}^1$ -bundle over  $S$  can be equipped with the additional structure of a principal  $\mathbb{G}_a$ -bundle. Furthermore, in this case, the set  $H^1(S, \mathbb{G}_m \times \mathbb{G}_a)$  of isomorphism classes of  $\mathbb{A}^1$ -bundles over  $S$  is isomorphic to the quotient of  $H^1(S, \mathbb{G}_a)$  by the action of  $H^0(S, \mathbb{G}_m) = \Gamma(S, \mathcal{O}_S^*)$  via a multiplicative reparametrization:  $a \in \Gamma(S, \mathcal{O}_S^*)$  sends the isomorphism class of the  $\mathbb{G}_a$ -bundle  $\rho : X \rightarrow S$  with action  $\mathbb{G}_{a,S} \times_S X \rightarrow X$ ,  $(t, x) \mapsto t \cdot x$  to the isomorphism class of  $\rho : X \rightarrow S$  equipped with the action  $(t, x) \mapsto (at) \cdot x$ .

**1.1.** It follows in particular that over the punctured affine plane  $\mathbb{A}_*^2$ , which has a trivial Picard group, the notions of  $\mathbb{A}^1$ -bundle and  $\mathbb{G}_a$ -bundle essentially coincide and that isomorphism classes of nontrivial  $\mathbb{A}^1$ -bundles are in one-to-one correspondence with elements of the infinite dimensional projective space  $\mathbb{P}H^1(\mathbb{A}_*^2, \mathcal{O}_{\mathbb{A}_*^2}) = H^1(\mathbb{A}_*^2, \mathcal{O}_{\mathbb{A}_*^2})/\mathbb{G}_m$ .

**1.2.** Every cohomology class in  $H^1(\mathbb{A}_*^2, \mathcal{O}_{\mathbb{A}_*^2})$  can be represented by a Čech 1-cocycle with value in  $\mathcal{O}_{\mathbb{A}_*^2}$  on the acyclic covering  $\mathcal{U}_0$  of  $\mathbb{A}_*^2 = \text{Spec}(\mathbb{C}[x, y]) \setminus \{0, 0\}$  by the principal affine open subsets  $U_x = \text{Spec}(\mathbb{C}[x^{\pm 1}, y])$  and  $U_y = \text{Spec}(\mathbb{C}[x, y^{\pm 1}])$ , providing an isomorphism of  $\mathbb{C}$ -vector spaces

$$H^1(\mathbb{A}_*^2, \mathcal{O}_{\mathbb{A}_*^2}) \simeq \check{H}^1(\mathcal{U}_0, \mathcal{O}_{\mathbb{A}_*^2}) \simeq \mathbb{C}[x^{\pm 1}, y^{\pm 1}] / \langle \mathbb{C}[x^{\pm 1}, y] + \mathbb{C}[x, y^{\pm 1}] \rangle.$$

It follows in particular from this description that  $H^1(\mathbb{A}_*^2, \mathcal{O}_{\mathbb{A}_*^2})$  is nonzero, which implies in turn that there exists nontrivial algebraic  $\mathbb{G}_a$ -bundles over  $\mathbb{A}_*^2$ . In contrast, in the differentiable category, all these bundles are globally trivial smooth fibrations with fibers  $\mathbb{R}^2$  over  $\mathbb{A}_*^2 \simeq \mathbb{R}^4 \setminus \{0\}$  equipped with its euclidean structure of differentiable manifold. Indeed, since the sheaf  $\mathcal{F} = \mathcal{C}^\infty(\mathbb{A}_*^2, \mathbb{C})$  of complex valued  $\mathcal{C}^\infty$ -functions on  $\mathbb{A}_*^2$  is soft (see e.g. [11, Theorem 5, p.25]), every algebraic Čech 1-cocycle  $g \in C^1(\mathcal{U}_0, \mathcal{O}_{\mathbb{A}_*^2})$  representing a nontrivial class in  $H^1(\mathbb{A}_*^2, \mathcal{O}_{\mathbb{A}_*^2})$  is a coboundary when considered as a 1-cocycle in with values in  $\mathcal{F}$ . This implies in particular that every algebraic  $\mathbb{G}_a$ -bundle  $\rho : X \rightarrow \mathbb{A}_*^2$  admits the real sphere  $S^3$  as a  $\mathcal{C}^\infty$ -strong deformation retract.

**Example 1.1.** A well known example of nontrivial algebraic  $\mathbb{G}_a$ -bundle over  $\mathbb{A}_*^2$  is given by the morphism

$$\rho : \text{SL}_2(\mathbb{C}) = \left\{ \begin{pmatrix} x & u \\ y & v \end{pmatrix} \in \mathcal{M}_2(\mathbb{C}), xv - yu = 1 \right\} \rightarrow \mathbb{A}_*^2, \begin{pmatrix} x & u \\ y & v \end{pmatrix} \mapsto (x, y)$$

which identifies  $\mathbb{A}_*^2$  with the quotient of  $\text{SL}_2(\mathbb{C})$  by the right action of its subgroup  $T \simeq \mathbb{G}_a$  of upper triangular matrices with 1's on the diagonal. The local trivializations of  $\rho$  given by the  $\mathbb{G}_a$ -equivariant isomorphisms  $\text{SL}_2(\mathbb{C})|_{U_x} \simeq U_x \times \text{Spec}(\mathbb{C}[x^{-1}u])$  and  $\text{SL}_2(\mathbb{C})|_{U_y} \simeq U_y \times \text{Spec}(\mathbb{C}[y^{-1}v])$  differ over  $U_x \cap U_y$  by the nontrivial Čech 1-cocycle  $(xy)^{-1} \in C^1(\mathcal{U}_0, \mathcal{O}_{\mathbb{A}_*^2}) = \mathbb{C}[x^{\pm 1}, y^{\pm 1}]$ . In contrast, the identity  $(xy)^{-1} = \delta(x, y)^{-1}(\overline{xy}^{-1} + x^{-1}\overline{y})$ , where  $\delta(x, y) = |x|^2 + |y|^2 \in \mathcal{C}^\infty(\mathbb{A}_*^2, \mathbb{R}_+^*)$ , shows that  $(xy)^{-1} \in C^1(\mathcal{U}_0, \mathcal{C}^\infty(\mathbb{A}_*^2, \mathbb{C}))$  is a coboundary.

## 1.2. Algebraic $\mathbb{G}_a$ -bundles with affine total spaces.

Since  $\mathbb{A}_*^2$  is strictly quasi-affine, the total space of a  $\mathbb{G}_a$ -bundle over it need not be an affine variety in general. However, we have the following handsome characterization of  $\mathbb{G}_a$ -bundles with affine total spaces.

**Proposition 1.2.** *A  $\mathbb{G}_a$ -bundle  $\rho : X \rightarrow \mathbb{A}_*^2$  has affine total space  $X$  if and only if it is nontrivial.*

*Proof.* See also [5] for an alternative argument. The condition is necessary since the total space of the trivial  $\mathbb{A}^1$ -bundle  $\mathbb{A}_*^2 \times \mathbb{A}^1$  is again strictly quasi-affine. Conversely, it is equivalent to show that if  $\rho : X \rightarrow \mathbb{A}_*^2$  is a nontrivial  $\mathbb{G}_a$ -bundle then the composition

$$i \circ \rho : X \rightarrow \mathbb{A}_*^2 \hookrightarrow \mathbb{A}^2 = \text{Spec}(\Gamma(\mathbb{A}_*^2, \mathcal{O}_{\mathbb{A}_*^2}))$$

is an affine morphism. Let  $o = (0, 0)$  be the origin of  $\mathbb{A}^2$  and  $S = \text{Spec}(\mathcal{O}_{\mathbb{A}^2, o}) \rightarrow \mathbb{A}^2$  be the natural morphism. Since  $\rho : X \rightarrow \mathbb{A}_*^2$  is an affine morphism,  $i \circ \rho$  is affine if and only if the base extension  $p_1 : S \times_{\mathbb{A}^2} X \rightarrow S$  is as well. It follows from a characterization due to Miyanishi [21] that either the restriction of  $p_1$  over the complement of the closed point  $o$  of  $S$  is a trivial  $\mathbb{A}^1$ -bundle or  $S \times_{\mathbb{A}^2} X$  is isomorphic to a closed subscheme of  $S \times \text{Spec}(\mathbb{C}[u, v])$  defined by an equation of the form  $fv - gu = 1$ , where  $f, g \in \mathcal{O}_{\mathbb{A}^2, o}$  is a regular sequence. In the second case,  $p_1^{-1}(S) = p_1^{-1}(S \setminus \{o\})$  is an affine scheme, and so  $p_1$  is an affine morphism. Otherwise, if  $p_1^{-1}(S \setminus \{o\}) \rightarrow S \setminus \{o\}$  is the trivial  $\mathbb{A}^1$ -bundle, then there exists an open subset  $U$  of  $\mathbb{A}^2$  containing  $o$  such that  $\rho^{-1}(U \setminus \{o\}) \rightarrow U \setminus \{o\}$  is a trivial  $\mathbb{G}_a$ -bundle. Therefore,  $\rho : X \rightarrow \mathbb{A}_*^2$  can be extended to a  $\mathbb{G}_a$ -bundle  $\bar{\rho} : \overline{X} \rightarrow \mathbb{A}^2$  over  $\mathbb{A}^2$ . But  $\mathbb{A}^2$  is affine, so  $\bar{\rho} : \overline{X} \rightarrow \mathbb{A}^2$  is a trivial  $\mathbb{G}_a$ -bundle, and as would be  $\rho : X \rightarrow \mathbb{A}_*^2$ , contradicting our hypothesis.  $\square$

**Example 1.3.** Every non trivial class in  $H^1(\mathbb{A}_*^2, \mathcal{O}_{\mathbb{A}_*^2})$  can be represented by a Čech 1-cocycle of the form  $x^{-m}y^{-n}p(x, y) \in C^1(\mathcal{U}_0, \mathcal{O}_{\mathbb{A}_*^2}) = \mathbb{C}[x^{\pm 1}, y^{\pm 1}]$ , where  $m, n \in \mathbb{N} \setminus \{0\}$  and  $p(x, y) \in \mathbb{C}[x, y]$  is a polynomial divisible neither by  $x$  nor by  $y$  and satisfying  $\deg_x p < m$  and  $\deg_y p < n$ . A corresponding  $\mathbb{A}^1$ -bundle  $\rho : X(m, n, p) \rightarrow \mathbb{A}_*^2$  is obtained as the complement in the variety

$$Z_{m,n,p} = \{x^m v - y^n u = p(x, y)\} \subset \mathbb{A}_*^2 \times \text{Spec}(\mathbb{C}[u, v])$$

of the fiber  $\text{pr}_{x,y}^{-1}|_{Z_{m,n,p}}(0, 0)$ . The latter is a  $\mathbb{G}_a$ -bundle when equipped for instance with the restriction of the  $\mathbb{G}_a$ -action  $t \cdot (x, y, u, v) = (x, y, u + x^m t, v + y^n t)$  on  $Z_{m,n,p}$ . Indeed, by construction of  $X(m, n, p)$ , with respect to the local  $\mathbb{G}_a$ -equivariant trivializations

$X(m, n, p)|_{U_x} \simeq U_x \times \text{Spec}(\mathbb{C}[x^{-m}u]) \simeq U_x \times \mathbb{G}_a$  and  $X(m, n, p)|_{U_y} \simeq U_y \times \text{Spec}(\mathbb{C}[y^{-n}v]) \simeq U_y \times \mathbb{G}_a$  the fiber coordinates differ precisely by the Čech 1-cocycle  $x^{-m}y^{-n}p(x, y) \in C^1(\mathcal{U}_0, \mathcal{O}_{\mathbb{A}_*^2})$ .

**1.3.** A consequence of Proposition 1.2 above is that the cylinders  $X \times \mathbb{A}^1$  over the total spaces of nontrivial  $\mathbb{G}_a$ -bundles  $\rho : X \rightarrow \mathbb{A}_*^2$  are not only all diffeomorphic in the euclidean topology but even all isomorphic as algebraic varieties. Indeed, given to such bundles  $X$  and  $X'$ , the fiber product  $X \times_{\mathbb{A}_*^2} X'$  is a  $\mathbb{G}_a$ -bundle over both  $X$  and  $X'$  via the first and the second projections respectively. Since  $X$  and  $X'$  are affine, the latter are both trivial as bundles over  $X$  and  $X'$  respectively providing isomorphisms  $X \times \mathbb{A}^1 \simeq X \times_{\mathbb{A}_*^2} X' \simeq X' \times \mathbb{A}^1$ . This implies in turn that most of the “standard” invariants of algebraic varieties which are either of topologico-differentiable nature or stable under taking cylinders fail to distinguish total spaces of nontrivial  $\mathbb{G}_a$ -bundles  $X \rightarrow \mathbb{A}_*^2$  considered as abstract affine varieties. For instance, algebraic vector bundles on the total space of a nontrivial  $\mathbb{A}^1$ -bundle  $\rho : X \rightarrow \mathbb{A}_*^2$  are all trivial. Indeed, this holds for  $\text{SL}_2(\mathbb{C})$  by virtue of [23] and the fact that vector bundles on  $X \times \mathbb{A}^1 \simeq \text{SL}_2(\mathbb{C}) \times \mathbb{A}^1$  are simultaneously extended from vector bundles on  $\text{SL}_2(\mathbb{C})$  and  $X$  [18] implies that this holds for arbitrary affine  $X$  too.

*Remark 1.4.* In the context of affine varieties  $V$  with  $\mathbb{G}_a$ -actions, the *Makar-Limanov invariant*  $\text{ML}(V)$ , defined as the algebra of regular functions on  $V$  that are invariant under *all* algebraic  $\mathbb{G}_a$ -actions on  $V$ , has been introduced and used by Makar-Limanov [19] to distinguish certain exotic algebraic structures on the affine 3-space. Clearly,  $\text{ML}(\text{SL}_2(\mathbb{C})) = \mathbb{C}$ , and since this invariant is known to be unstable under taking cylinders [2] [4], it is a natural candidate for distinguishing certain  $\mathbb{A}^1$ -bundles  $\rho : X \rightarrow \mathbb{A}_*^2$  from  $\text{SL}_2(\mathbb{C})$ . However, one checks easily using the explicit description in 1.3 above that  $\text{ML}(X) = \mathbb{C}$  for every non trivial  $\mathbb{A}^1$ -bundle  $\rho : X \rightarrow \mathbb{A}_*^2$ . Actually, if  $X = X(m, n, p)$ , where  $p \in \mathbb{C}[x, y]$  is a homogeneous polynomial, then Theorem 2.3 below implies that the total space of  $X$  is isomorphic to  $X_{1,r} = \{xv - y^r u = 1\}$ , where  $r + 1 = m + n - \deg p \geq 2$ , which is even a *flexible variety*, i.e., the tangent space at every point of  $x$  is spanned by the tangent vectors to the orbits of the  $\mathbb{G}_a$ -actions on  $X$  [1]. We do not know if this additional property holds for general  $\mathbb{A}^1$ -bundles  $\rho : X \rightarrow \mathbb{A}_*^2$ .

### 1.3. Homogeneous $\mathbb{G}_a$ -bundles.

Here we develop the notion of homogeneous  $\mathbb{G}_a$ -bundle over  $\mathbb{A}_*^2$  that will be used in the rest of the article.

**1.4.** Let  $\sigma : \mathbb{G}_m \times \mathbb{A}_*^2 \rightarrow \mathbb{A}_*^2$  denote the linear  $\mathbb{G}_m$ -action on  $\mathbb{A}_*^2$  with quotient  $\pi : \mathbb{A}_*^2 \rightarrow \mathbb{P}^1 = \text{Proj}(\mathbb{C}[x, y])$ . Since  $\pi$  is an affine morphism and  $\pi_* \mathcal{O}_{\mathbb{A}_*^2} \simeq \bigoplus_{d \in \mathbb{Z}} \mathcal{O}_{\mathbb{P}^1}(-d)$ , we have a decomposition

$$H^1(\mathbb{A}_*^2, \mathcal{O}_{\mathbb{A}_*^2}) \simeq H^1(\mathbb{P}^1, \pi_* \mathcal{O}_{\mathbb{A}_*^2}) \simeq \bigoplus_{d \in \mathbb{Z}} H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-d)) \simeq \bigoplus_{d \geq 2} H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-d)),$$

where for every  $d$ ,  $H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-d))$  can be identified with the vector space of the semi-invariants of weight  $-d$  for the representation of  $\mathbb{G}_m$  on  $H^1(\mathbb{A}_*^2, \mathcal{O}_{\mathbb{A}_*^2})$  induced by  $\sigma$ . Via the one-to-one correspondence between coverings  $\mathcal{V} = (V_i)_{i \in I}$  of  $\mathbb{P}^1 = \mathbb{A}_*^2/\mathbb{G}_m$  by affine open subsets  $V_i$ ,  $i \in I$  and coverings  $\mathcal{U} = (U_i)_{i \in I}$  of  $\mathbb{A}_*^2$  by  $\mathbb{G}_m$ -stable affine open subsets  $U_i = \pi^{-1}(V_i)$ ,  $i \in I$ , a cohomology class in  $H^1(\mathbb{P}^1, \pi_* \mathcal{O}_{\mathbb{A}_*^2})$  belongs to  $H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-d))$  if and only if it can be represented by a Čech 1-cocycle  $\{h_{ij}\}_{i,j \in I} \in C^1(\mathcal{V}, \mathcal{O}_{\mathbb{P}^1}(-d)) \subset C^1(\mathcal{V}, \pi_* \mathcal{O}_{\mathbb{A}_*^2}) \simeq C^1(\mathcal{U}, \mathcal{O}_{\mathbb{A}_*^2})$  consisting of rational functions  $h_{ij} \in \Gamma(U_i \cap U_j, \mathcal{O}_{\mathbb{A}_*^2}) \subset \mathbb{C}(x, y)$  that are homogeneous of degree  $-d$ . These cocycles correspond precisely to  $\mathbb{G}_a$ -bundles  $\tilde{\rho} : \tilde{X} \rightarrow \mathbb{A}_*^2$  with local trivializations  $\tau_i : \tilde{X}|_{U_i} \xrightarrow{\sim} U_i \times \mathbb{G}_a$  for which the isomorphisms  $\tau_i \circ \tau_j^{-1}|_{U_i \cap U_j}, (u, t_j) \mapsto (u, t_j + h_{ij}(u))$ ,  $i, j \in I$ , are equivariant for the actions of  $\mathbb{G}_m$  on  $U_i \times \mathbb{G}_a$  and  $U_j \times \mathbb{G}_a$  by  $\lambda \cdot (u, t) = (\sigma(\lambda, u), \lambda^{-d}t)$ . This leads to the following interpretation of the above decomposition in terms of  $\mathbb{G}_a$ -bundles over  $\mathbb{A}_*^2$ .

**Proposition 1.5.** *For a  $\mathbb{G}_a$ -bundle  $\rho : X \rightarrow \mathbb{A}_*^2$ , the following are equivalent:*

- a) *The isomorphism class of  $\rho : X \rightarrow \mathbb{A}_*^2$  belongs to  $H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-d)) \subset H^1(\mathbb{A}_*^2, \mathcal{O}_{\mathbb{A}_*^2})$ ,*
- b)  *$\rho : X \rightarrow \mathbb{A}_*^2$  is isomorphic to a  $\mathbb{G}_a$ -bundle  $\tilde{\rho} : \tilde{X} \rightarrow \mathbb{A}_*^2$  admitting a lift  $\tilde{\sigma} : \mathbb{G}_m \times \tilde{X} \rightarrow \tilde{X}$  of  $\sigma$  for which there exists a collection of  $\mathbb{G}_a$ -equivariant local trivializations  $\tau_i : \tilde{X}|_{U_i} \xrightarrow{\sim} U_i \times \mathbb{G}_a$  over a covering of  $\mathbb{A}_*^2$  by  $\mathbb{G}_m$ -stable affine open subsets  $(U_i)_{i \in I}$  such that for every  $i \in I$ ,  $\tau_i$  is  $\mathbb{G}_m$ -equivariant for the action of  $\mathbb{G}_m$  on  $U_i \times \mathbb{G}_a$  defined by  $\lambda \cdot (u, t) = (\sigma(\lambda, u), \lambda^{-d}t)$ .*

**Definition 1.6.** A nontrivial  $\mathbb{G}_a$ -bundle  $\rho : X \rightarrow \mathbb{A}_*^2$  satisfying one of the above equivalent properties for a certain  $d \geq 2$  is said to be  $-d$ -homogeneous.

**Example 1.7.** By specializing to the covering  $\mathcal{U}_0$  of  $\mathbb{A}_*^2$  by the  $\mathbb{G}_m$ -stable principal open subsets  $U_x$  and  $U_y$ , we obtain a more explicit decomposition:

$$H^1(\mathbb{A}_*^2, \mathcal{O}_{\mathbb{A}_*^2}) \simeq \check{H}^1(\mathcal{U}_0, \mathcal{O}_{\mathbb{A}_*^2}) \simeq \bigoplus_{d \geq 2} W_{-d}$$

where, for every  $d \geq 2$ ,  $W_{-d} \simeq H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-d))$  denotes the sub- $\mathbb{C}$ -vector space of  $\mathbb{C}[x^{\pm 1}, y^{\pm 1}]$  with basis  $\mathcal{B}_d$  consisting of rational monomials  $x^{-m}y^{-n}$  where  $m, n \in \mathbb{N} \setminus \{0\}$  and  $m+n = d$ . Note that letting  $V_{d-2} \simeq H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(d-2))$  be the space of binary forms of degree  $d-2$ , Serre duality for  $\mathbb{P}^1$  takes the form of a perfect pairing  $W_{-d} \times V_{d-2} \rightarrow W_{-2}$  for which the basis  $\mathcal{B}_d$  is simply the dual of the usual basis of  $V_{d-2}$  consisting of monomials  $x^p y^q$  with  $p+q = d-2$ . Therefore, every non trivial class in  $H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-d))$  is represented by a Čech 1-cocycle of the form  $x^{-m}y^{-n}p(x, y) \in C^1(\mathcal{U}_0, \mathcal{O}_{\mathbb{A}_*^2})$ , where  $p(x, y) \in \mathbb{C}[x, y]$  is a homogeneous polynomial of degree  $r = m+n-d \geq 0$ . The corresponding  $\mathbb{G}_a$ -bundle  $\tilde{X} = X(m, n, p) = \{x^m v - y^n u = p(x, y)\} \setminus \{x=y=0\}$  as in 1.3 admits an obvious lift  $\tilde{\sigma}(\lambda, (x, y, u, v)) = (\lambda x, \lambda y, \lambda^{m-d}u, \lambda^{n-d}v)$  of the  $\mathbb{G}_m$ -action  $\sigma$  on  $\mathbb{A}_*^2$  which satisfies b) in Proposition 1.5 above.

**1.5.** Proposition 1.5 can be interpreted from another point of view as a correspondence between  $-d$ -homogeneous  $\mathbb{G}_a$ -bundles  $\rho : X \rightarrow \mathbb{A}_*^2$  and *principal homogeneous bundles*  $\nu : Y \rightarrow \mathbb{P}^1$  under the line bundle  $p : \mathcal{O}_{\mathbb{P}^1}(-d) \rightarrow \mathbb{P}^1$ . Here we consider the line bundle  $p : \mathcal{O}_{\mathbb{P}^1}(-d) \rightarrow \mathbb{P}^1$  as equipped with the structure of a locally constant group scheme over  $\mathbb{P}^1$ , with group law induced by the diagonal homomorphism of sheaves  $\mathcal{O}_{\mathbb{P}^1}(d) \rightarrow \mathcal{O}_{\mathbb{P}^1}(d) \oplus \mathcal{O}_{\mathbb{P}^1}(d)$ . A principal homogeneous  $\mathcal{O}_{\mathbb{P}^1}(-d)$ -bundle (or simply an  $\mathcal{O}_{\mathbb{P}^1}(-d)$ -bundle) is a scheme  $\nu : Y \rightarrow \mathbb{P}^1$  equipped with an action of  $\mathcal{O}_{\mathbb{P}^1}(-d)$  such that every point of  $\mathbb{P}^1$  has a Zariski open neighborhood  $U$  such that  $Y|_U$  is equivariantly isomorphic to  $\mathcal{O}_{\mathbb{P}^1}(-d)|_U$  acting on itself by translations. Isomorphism classes of  $\mathcal{O}_{\mathbb{P}^1}(-d)$ -bundles are in one-to-one correspondence with elements of the group  $H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-d))$ . Example 1.8 below shows that for a  $\mathbb{G}_a$ -bundle  $\tilde{\rho} : \tilde{X} \rightarrow \mathbb{A}_*^2$  equipped with a lift  $\tilde{\sigma}$  of  $\sigma$  as in b), the quotient  $\mathbb{A}^1$ -bundle  $\nu : \tilde{X}/\mathbb{G}_m \rightarrow \mathbb{P}^1 = \mathbb{A}_*^2/\mathbb{G}_m$  comes naturally equipped with the structure of principal homogeneous  $\mathcal{O}_{\mathbb{P}^1}(-d)$ -bundle with isomorphism class  $\gamma = [\tilde{X}] \in H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-d)) \subset H^1(\mathbb{A}_*^2, \mathcal{O}_{\mathbb{A}_*^2})$ .

**Example 1.8.** By virtue of example 1.7 above, every nontrivial  $-d$ -homogeneous  $\mathbb{G}_a$ -bundle is isomorphic to one of the form  $\tilde{X} = X(m, n, p)$ , where  $p(x, y) \in \mathbb{C}[x, y]$  is homogeneous of degree  $r = m+n-d \geq 0$ . The latter is equipped with the lift  $\tilde{\sigma}(\lambda, (x, y, u, v)) = (\lambda x, \lambda y, \lambda^{m-d}u, \lambda^{n-d}v)$  of  $\sigma$  for which we have local  $\mathbb{G}_m$ -equivariant trivializations

$$\begin{aligned} \tilde{X}|_{U_x} &\simeq \text{Spec}(\mathbb{C}[x^{-1}y, x^{d-m}u]) \times \mathbb{G}_m = \text{Spec}(\mathbb{C}[z, w]) \times \mathbb{G}_m \\ \tilde{X}|_{U_y} &\simeq \text{Spec}(\mathbb{C}[xy^{-1}, y^{d-n}v]) \times \mathbb{G}_m = \text{Spec}(\mathbb{C}[z', w']) \times \mathbb{G}_m. \end{aligned}$$

These induce trivializations  $\tau_x : \nu^{-1}(U_x/\mathbb{G}_m) \xrightarrow{\sim} \text{Spec}(\mathbb{C}[z][w])$  and  $\tau_y : \nu^{-1}(U_y/\mathbb{G}_m) \xrightarrow{\sim} \text{Spec}(\mathbb{C}[z'][w'])$  of the quotient bundle  $\nu : \tilde{X}/\mathbb{G}_m \rightarrow \mathbb{P}^1$  for which the transition isomorphism  $\tau_y \circ \tau_x^{-1}|_{U_x \cap U_y/\mathbb{G}_m}$  has the form  $(z, w) \mapsto (z^{-1}, z^d w + z^m p(z^{-1}, 1))$ . To see explicitly the structure of an  $\mathcal{O}_{\mathbb{P}^1}(-d)$  bundle on  $\tilde{X}/\mathbb{G}_m$ , choose coordinates for the local trivializations of the total space of  $\mathcal{O}_{\mathbb{P}^1}(-d)$  as follows :

$$\begin{aligned} \mathcal{O}_{\mathbb{P}^1}(-d)|_{U_x/\mathbb{G}_m} &\simeq \text{Spec}(\mathbb{C}[x^{-1}y, x^d t]) = \text{Spec}(\mathbb{C}[z, \ell]) \\ \mathcal{O}_{\mathbb{P}^1}(-d)|_{U_y/\mathbb{G}_m} &\simeq \text{Spec}(\mathbb{C}[xy^{-1}, y^d t]) = \text{Spec}(\mathbb{C}[z', \ell']). \end{aligned}$$

Then we see that the  $\mathbb{G}_a$ -action  $t \cdot (x, y, u, v) = (x, y, u + x^m t, v + y^n t)$  on  $\tilde{X}$  descends to the action of  $\mathcal{O}_{\mathbb{P}^1}(-d)$  on  $\tilde{X}/\mathbb{G}_m$  defined locally by  $\ell \cdot w = w + \ell$  and  $\ell' \cdot w' = w' + \ell'$ , which equips  $\tilde{X}/\mathbb{G}_m$  with the structure of an  $\mathcal{O}_{\mathbb{P}^1}(-d)$ -bundle. Finally, with our choice of coordinate, the natural isomorphism of  $\mathbb{C}$ -vector spaces

$$\phi : C^1(\{U_x, U_y\}, \mathcal{O}_{\mathbb{A}_*^2}) = \mathbb{C}[x^{\pm 1}, y^{\pm 1}] \xrightarrow{\sim} C^1(\{U_x/\mathbb{G}_m, U_y/\mathbb{G}_m\}, \pi_* \mathcal{O}_{\mathbb{A}_*^2}) = \mathbb{C}[z^{\pm 1}]$$

maps a Laurent monomial  $x^i y^j$  to  $z^{-i}$ , whence sends the Čech cocycle  $x^{-m}y^{-n}p(x, y)$  representing the isomorphism class of the  $\mathbb{G}_a$ -bundle  $\tilde{\rho} : \tilde{X} \rightarrow \mathbb{A}_*^2$  to the one  $z^m p(z^{-1}, 1)$  representing the isomorphism class of the  $\mathcal{O}_{\mathbb{P}^1}(-d)$ -bundle  $\nu : \tilde{X}/\mathbb{G}_m \rightarrow \mathbb{P}^1$ .

**Remark 1.9.** The precise correspondence between  $-d$ -homogeneous  $\mathbb{G}_a$ -bundles over  $\mathbb{A}_*^2$  and principal  $\mathcal{O}_{\mathbb{P}^1}(-d)$ -bundles takes the form of an equivalence of categories extending the one between  $\mathbb{G}_m$ -linearized line bundles on  $\mathbb{A}_*^2$  with respect to the action  $\sigma : \mathbb{G}_m \times \mathbb{A}_*^2 \rightarrow \mathbb{A}_*^2$  and line bundles over  $\mathbb{P}^1 = \mathbb{A}_*^2/\mathbb{G}_m$ . Recall that a  $\mathbb{G}_m$ -linearized line bundle is a pair  $(L, \Phi)$  consisting of a line bundle  $p : L \rightarrow \mathbb{A}_*^2$  and an isomorphism

$$\Phi : \sigma^* L = (\mathbb{G}_m \times \mathbb{A}_*^2) \times_{\sigma, \mathbb{A}_*^2} L \xrightarrow{\sim} p^* L = (\mathbb{G}_m \times \mathbb{A}_*^2) \times_{p, \mathbb{A}_*^2} L$$

of line bundles over  $\mathbb{G}_m \times \mathbb{A}_*^2$  satisfying the cocycle condition  $(\mu \times \text{id}_{\mathbb{A}_*^2})^* \Phi = p_{23}^* \Phi \circ (\text{id}_{\mathbb{G}_m} \times \sigma)^* \Phi$  over  $\mathbb{G}_m \times \mathbb{G}_m \times \mathbb{A}_*^2$ , where  $\mu : \mathbb{G}_m \times \mathbb{G}_m \rightarrow \mathbb{G}_m$  denotes the group law of  $\mathbb{G}_m$  (see e.g. [22, §3, p.30]). A standard argument of faithfully flat descent for the quotient morphism  $\pi : \mathbb{A}_*^2 \rightarrow \mathbb{P}^1 = \mathbb{A}_*^2 / \mathbb{G}_m$  shows that the category of  $\mathbb{G}_m$ -linearized line bundles over  $\mathbb{A}_*^2$  is equivalent to category of line bundles over  $\mathbb{P}^1$ . Noting that  $\Phi : \sigma^* L \xrightarrow{\sim} p_2^* L$  is an isomorphism of group schemes over  $\mathbb{G}_m \times \mathbb{A}_*^2$ , we can define a category  $\tilde{\mathbb{V}}$  whose objects are pairs  $\{(L, \Phi), (\tilde{X}, \Psi)\}$  consisting of a  $\mathbb{G}_m$ -linearized line bundle  $(L, \Phi)$ , a principal  $L$ -bundle  $\tilde{\rho} : \tilde{X} \rightarrow \mathbb{A}_*^2$ , and a  $\Phi$ -equivariant isomorphism  $\Psi : \sigma^* \tilde{X} \xrightarrow{\sim} p_2^* \tilde{X}$  of principal bundles under  $\sigma^* L$  and  $p_2^* L$  respectively, satisfying the cocycle condition  $(\mu \times \text{id}_{\mathbb{A}_*^2})^* \Psi = p_{23}^* \Psi \circ (\text{id}_{\mathbb{G}_m} \times \sigma)^* \Psi$  over  $\mathbb{G}_m \times \mathbb{G}_m \times \mathbb{A}_*^2$ . Then one checks that the previous equivalence extends to a one between  $\tilde{\mathbb{V}}$  and the category  $\mathbb{V}$  whose objects are pairs  $(M, Y)$  consisting of a line bundle  $q : M \rightarrow \mathbb{P}^1$  and a principal  $M$ -bundle  $\nu : Y \rightarrow \mathbb{P}^1$ .

Recall that for a  $\mathbb{G}_m$ -linearized line bundle  $(L, \Phi)$  over  $\mathbb{A}_*^2$  the morphism  $\bar{\sigma} = p_2 \circ \Phi^{-1} : \mathbb{G}_m \times L \simeq \sigma^* L \rightarrow L$  defines a lift to  $L$  of the  $\mathbb{G}_m$ -action  $\sigma$  on  $\mathbb{A}_*^2$  which is “linear on the fibers” of  $p : L \rightarrow \mathbb{A}_*^2$ . Similarly, for  $(\tilde{X}, \Psi)$  as above, the morphism  $\tilde{\sigma} = p_2 \circ \Psi^{-1} : \mathbb{G}_m \times \tilde{X} \simeq \sigma^* \tilde{X} \rightarrow \tilde{X}$  is a lift to  $\tilde{X}$  of  $\sigma$  for which  $\tilde{X}$  “locally looks like  $L$  equipped with the action  $\bar{\sigma}$ ”. By specializing to the case of the trivial line bundle  $\mathbb{A}_*^2 \times \text{Spec}(\mathbb{C}[t])$  over  $\mathbb{A}_*^2$  equipped with the  $\mathbb{G}_m$ -linearization given by the lift  $\lambda \cdot (x, y, \ell) = (\lambda x, \lambda y, \lambda^{-d} \ell)$  of  $\sigma$ , which corresponds to the line bundle  $\mathcal{O}_{\mathbb{P}^1}(-d) \rightarrow \mathbb{P}^1$ , the previous equivalence boils down to a one-to-one correspondence between isomorphism classes of principal  $\mathcal{O}_{\mathbb{P}^1}(-d)$ -bundles over  $\mathbb{P}^1$  and isomorphism classes of  $\mathbb{G}_a$ -bundles  $\tilde{\rho} : \tilde{X} \rightarrow \mathbb{A}_*^2$  equipped with a lift  $\tilde{\sigma} : \mathbb{G}_m \times \tilde{X} \rightarrow \tilde{X}$  of  $\sigma$  as in b) in Proposition 1.5.

## 2. ISOMORPHY TYPES OF TOTAL SPACES OF $\mathbb{A}^1$ -BUNDLES OVER $\mathbb{A}_*^2$

In this section, we give partial answers to the problem of classifying total spaces of nontrivial  $\mathbb{A}^1$ -bundles over  $\mathbb{A}_*^2$  considered as abstract affine varieties.

### 2.1. Base change under the action of $\text{Aut}(\mathbb{A}_*^2)$ .

Since we are interested in isomorphism types of total spaces of  $\mathbb{A}^1$ -bundles over  $\mathbb{A}_*^2$  as abstract varieties, regardless of the particular  $\mathbb{A}^1$ -bundle structure, a natural step is to consider these bundles up to a weaker notion of bundle isomorphism which consists in identifying two nontrivial  $\mathbb{A}^1$ -bundles  $\rho : X \rightarrow \mathbb{A}_*^2$  and  $\rho' : X' \rightarrow \mathbb{A}_*^2$  if there exists a commutative diagram

$$\begin{array}{ccc} X' & \xrightarrow{\Psi} & X \\ \rho' \downarrow & & \downarrow \rho \\ \mathbb{A}_*^2 & \xrightarrow{\psi} & \mathbb{A}_*^2 \end{array}$$

where  $\psi$  and  $\Psi$  are isomorphisms. This means equivalently that the isomorphism classes of  $X$  and  $X'$  in  $\mathbb{P}H^1(\mathbb{A}_*^2, \mathcal{O}_{\mathbb{A}_*^2})$  belong to the same orbit of the action of the group  $\text{Aut}(\mathbb{A}_*^2)$  of automorphisms of  $\mathbb{A}_*^2$  on  $\mathbb{P}H^1(\mathbb{A}_*^2, \mathcal{O}_{\mathbb{A}_*^2})$  induced by the linear representation

$$\eta : \text{Aut}(\mathbb{A}_*^2) \rightarrow \text{GL}(H^1(\mathbb{A}_*^2, \mathcal{O}_{\mathbb{A}_*^2})), \quad \psi \mapsto \eta(\psi) = \psi^* : H^1(\mathbb{A}_*^2, \mathcal{O}_{\mathbb{A}_*^2}) \xrightarrow{\sim} H^1(\mathbb{A}_*^2, \mathcal{O}_{\mathbb{A}_*^2})$$

of  $\text{Aut}(\mathbb{A}_*^2)$  on  $H^1(\mathbb{A}_*^2, \mathcal{O}_{\mathbb{A}_*^2})$ , where  $\psi^*$  maps the isomorphism class of  $\mathbb{G}_a$ -bundle  $\rho : X \rightarrow \mathbb{A}_*^2$  to that of the  $\mathbb{G}_a$ -bundle  $\text{pr}_2 : X \times_{\rho, \mathbb{A}_*^2, \psi} \mathbb{A}_*^2 \rightarrow \mathbb{A}_*^2$ .

**2.1.** The group of automorphisms of  $\mathbb{A}_*^2$  can be identified with the subgroup of  $\text{Aut}(\mathbb{A}^2)$  consisting of automorphisms of the plane  $\mathbb{A}^2$  that preserve the origin  $o$ . As a consequence of Jung’s Theorem [16],  $\text{Aut}(\mathbb{A}_*^2)$  is generated by the general linear group  $\text{GL}_2(\mathbb{C})$  and the subgroup  $U \subset \text{Aut}(\mathbb{A}_*^2)$  consisting of automorphisms of the form  $(x, y) \mapsto (x, y + p(x))$  where  $p(x) \in x^2 \mathbb{C}[x]$ . Since the representation of  $\mathbb{G}_m$  on  $H^1(\mathbb{A}_*^2, \mathcal{O}_{\mathbb{A}_*^2})$  induced by the action  $\sigma : \mathbb{G}_m \times \mathbb{A}_*^2 \rightarrow \mathbb{A}_*^2$  commutes with that of  $\text{GL}_2(\mathbb{C})$ , the decomposition

$$H^1(\mathbb{A}_*^2, \mathcal{O}_{\mathbb{A}_*^2}) \simeq \bigoplus_{d \geq 2} H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-d))$$

provides a splitting of the induced representation  $\text{GL}_2(\mathbb{C}) \rightarrow \text{GL}(H^1(\mathbb{A}_*^2, \mathcal{O}_{\mathbb{A}_*^2}))$  into a direct sum of representations on the finite dimensional vector spaces  $H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-d))$ ,  $d \geq 2$ . Using the identifications  $H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-d)) \simeq W_{-d}$  and  $H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(d-2)) \simeq V_{d-2}$  as in example 1.7, the perfect pairing  $W_{-d} \times V_{d-2} \rightarrow W_{-2}$  given by Serre duality for  $\mathbb{P}^1$  yields an isomorphism of representations  $W_{-d} \simeq V_{d-2}^* \otimes W_{-2}$ , where  $\text{GL}_2(\mathbb{C})$  acts on the vector space  $V_{d-2}$  of binary forms of degree  $d-2$  via the standard representation and on  $W_{-2} \simeq H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-2)) \simeq \mathbb{C}$  by the inverse of the determinant.

**2.2.** Since triangular automorphisms in  $U$  do not preserve the usual degree on  $\mathbb{C}[x, y]$ , the induced representation  $U \rightarrow \text{GL}(H^1(\mathbb{A}_*^2, \mathcal{O}_{\mathbb{A}_*^2}))$  does not preserve the above decomposition of  $H^1(\mathbb{A}_*^2, \mathcal{O}_{\mathbb{A}_*^2})$  into a direct sum. However, letting  $F_{-d} = \bigoplus_{i=2}^d W_{-i} \subset H^1(\mathbb{A}_*^2, \mathcal{O}_{\mathbb{A}_*^2})$ ,  $d \geq 2$ , we have the following description.

**Lemma 2.1.** *For every  $d \geq 2$ , the subspace  $F_{-d}$  is  $U$ -stable and the quotient representation on  $F_{-(d+1)}/F_{-d} \simeq W_{-(d+1)}$  is the trivial one.*

*Proof.* It is equivalent to show that the subspace  $Q_{-d}^*$  of the dual of  $H^1(\mathbb{A}_*^2, \mathcal{O}_{\mathbb{A}_*^2})$  that is orthogonal to  $F_{-d}$  is stable under the dual representation, and that the quotient representation on  $Q_{-d}^*/Q_{-(d+1)}^*$  is the trivial one. By Serre duality again, we have

$$(H^1(\mathbb{A}_*^2, \mathcal{O}_{\mathbb{A}_*^2}))^* \simeq (\bigoplus_{i \geq 2} W_{-i})^* \simeq \prod_{i \geq 2} W_{-i}^* \simeq \prod_{i \geq 2} V_{i-2},$$

the dual representation on  $\prod_{i \geq 2} V_{i-2}$  being induced by the action of  $U$  on  $\mathbb{C}[x, y]$  defined by  $u \cdot p(x, y) = p(u^{-1}(x, y))$ . For an element  $u = (x, y + x^2 s(x)) \in U$  and a homogeneous polynomial  $p_n(x, y) \in V_n$  of degree  $n \geq 0$ , one has  $u \cdot p_n(x, y) = p_n(x, y) + R(x, y)$ , where  $R$  is a finite sum of homogeneous polynomials of degrees  $> n$ . This implies that

$$Q_{-d}^* = \prod_{i > d} V_{i-2} \subset \prod_{i \geq 2} V_{i-2}$$

is  $U$ -stable and that the quotient representation on  $Q_{-d}^*/Q_{-(d+1)}^* \simeq V_{d-2}$  is the trivial one, as desired.  $\square$

**2.3.** One cannot expect to have a general effective criterion to decide which isomorphism classes of  $\mathbb{G}_a$ -bundles or  $\mathbb{A}^1$ -bundles over  $\mathbb{A}_*^2$  belong to the same orbit of the actions of  $\text{Aut}(\mathbb{A}_*^2)$  on  $H^1(\mathbb{A}_*^2, \mathcal{O}_{\mathbb{A}_*^2})$  and  $\mathbb{P}H^1(\mathbb{A}_*^2, \mathcal{O}_{\mathbb{A}_*^2})$  respectively. But the above description provides at least strong restrictions for certain homogeneous  $\mathbb{G}_a$ -bundles to be obtained as pull-backs of other ones by an automorphism of  $\mathbb{A}_*^2$ . For instance, the isomorphism class of the  $\mathbb{A}^1$ -bundle  $\text{pr}_{x,y} : \text{SL}_2(\mathbb{C}) = \{xv - yu = 1\} \rightarrow \mathbb{A}_*^2$  is a fixed point of the projective representation of  $\text{Aut}(\mathbb{A}_*^2)$  on  $\mathbb{P}H^1(\mathbb{A}_*^2, \mathcal{O}_{\mathbb{A}_*^2})$ , whence is stable under arbitrary base change by an automorphism of  $\mathbb{A}_*^2$ . In the same spirit, for the isomorphism classes of the homogeneous  $\mathbb{G}_a$ -bundles

$$\text{pr}_{x,y} : X_{m,n} = X(m, n, 1) = \{x^m v - y^n u = 1\} \rightarrow \mathbb{A}_*^2,$$

we have following result (compare with Theorem 2.3 and example 2.4 below).

**Proposition 2.2.** *The  $\mathbb{A}^1$ -bundles  $X_{m,n} \rightarrow \mathbb{A}_*^2$  and  $X_{p,q} \rightarrow \mathbb{A}_*^2$  can be obtained from each other by a base change  $\psi : \mathbb{A}_*^2 \xrightarrow{\sim} \mathbb{A}_*^2$  if and only if  $\{m, n\} = \{p, q\}$ .*

*Proof.* It is equivalent to show that for  $a \in \mathbb{C}^*$  the isomorphism classes in  $H^1(\mathbb{A}_*^2, \mathcal{O}_{\mathbb{A}_*^2})$  of  $X_{p,q}$  and  $X_{m,n}(a) = X(m, n, a)$  belong to the same orbit of the action of  $\text{Aut}(\mathbb{A}_*^2)$  if and only if  $\{m, n\} = \{p, q\}$ . Since  $X_{m,n}(a)$  and  $X_{p,q}$  are homogeneous of degree  $-m - n$  and  $-p - q$ , it follows from Proposition 1.5 and Lemma 2.1 that their isomorphism classes  $[X_{m,n}(a)] \in W_{-(m+n)}$  and  $[X_{p,q}] \in W_{-(p+q)}$  belong to the same orbit of  $\text{Aut}(\mathbb{A}_*^2)$  if and only if  $m + n = p + q = d$  and they belong to the same orbit of the action of  $\text{GL}_2(\mathbb{C})$  on  $W_{-d}$ . By duality, this holds if and only if the homogeneous polynomials  $a^{-1}x^{m-1}y^{n-1}$  and  $x^{p-1}y^{q-1}$  belong to the same orbit of the standard representation of  $\text{GL}_2(\mathbb{C})$  on  $V_{d-2}$ , which is the case if and only if  $\{m-1, n-1\} = \{p-1, q-1\}$ .  $\square$

## 2.2. Isomorphism types of homogeneous $\mathbb{G}_a$ -bundles.

Recall 1.3 that a nontrivial  $\mathbb{G}_a$ -bundle  $\rho : X \rightarrow \mathbb{A}_*^2$  is called  $-d$ -homogeneous if it represents a cohomology class in  $H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-d)) \subset H^1(\mathbb{A}_*^2, \mathcal{O}_{\mathbb{A}_*^2})$  for a certain  $d \geq 2$ . All these bundles come equipped with a lift of the  $\mathbb{G}_m$ -action  $\sigma : \mathbb{G}_m \times \mathbb{A}_*^2 \rightarrow \mathbb{A}_*^2$  as in Proposition 1.5, and we have the following characterization:

**Theorem 2.3.** *The total spaces of two nontrivial homogeneous  $\mathbb{G}_a$ -bundles are  $\mathbb{G}_m$ -equivariantly isomorphic if and only if they have the same homogeneous degree. In particular, for a fixed  $d \geq 2$ , the total spaces of nontrivial  $-d$ -homogeneous  $\mathbb{G}_a$ -bundles  $\rho : X \rightarrow \mathbb{A}_*^2$  are all isomorphic as abstract affine varieties.*

*Proof.* Suppose that  $\rho : X \rightarrow \mathbb{A}_*^2$  and  $\rho' : X' \rightarrow \mathbb{A}_*^2$  are homogeneous of degrees  $-d$  and  $-d'$  respectively. A  $\mathbb{G}_m$ -equivariant isomorphism between  $X'$  and  $X$  descends to an isomorphism  $f : X'/\mathbb{G}_m \xrightarrow{\sim} X/\mathbb{G}_m$  between the total space of the principal  $\mathcal{O}_{\mathbb{P}^1}(-d)$ -bundle  $\bar{\rho} : X/\mathbb{G}_m \rightarrow \mathbb{P}^1$  and that of the principal  $\mathcal{O}_{\mathbb{P}^1}(-d')$ -bundle  $\bar{\rho}' : X'/\mathbb{G}_m \rightarrow \mathbb{P}^1$  (see 1.5). The Picard group of a principal  $\mathcal{O}_{\mathbb{P}^1}(-i)$ -bundle  $\nu : Y \rightarrow \mathbb{P}^1$  is isomorphic to  $\mathbb{Z}$ , generated for instance by the pull-back  $\nu^* \mathcal{O}_{\mathbb{P}^1}(-1)$  of  $\mathcal{O}_{\mathbb{P}^1}(-1)$ . Furthermore it follows from the exact sequence

$$0 \rightarrow \nu^* \Omega_{\mathbb{P}^1}^1 \rightarrow \Omega_Y^1 \rightarrow \Omega_{Y/\mathbb{P}^1}^1 \simeq \nu^* \mathcal{O}_{\mathbb{P}^1}(i) \rightarrow 0$$

that  $\omega_Y = \Lambda^2 \Omega_Y^1 \simeq \nu^* \mathcal{O}_{\mathbb{P}^1}(i-2)$ . Since the isomorphism  $f^* : \text{Pic}(X/\mathbb{G}_m) \xrightarrow{\sim} \text{Pic}(X'/\mathbb{G}_m)$  induced by  $f$  sends  $\omega_{X/\mathbb{G}_m}$  to  $\omega_{X'/\mathbb{G}_m}$ , we conclude that  $d = d'$  necessarily. Suppose conversely that  $\rho : X \rightarrow \mathbb{A}_*^2$  and  $\rho' : X' \rightarrow \mathbb{A}_*^2$  are homogeneous of the same degree  $-d \leq -2$ . The existence of a  $\mathbb{G}_m$ -equivariant isomorphism between  $X'$  and  $X$  is equivalent to the existence of an isomorphism  $f : X'/\mathbb{G}_m \xrightarrow{\sim} X/\mathbb{G}_m$

for which  $\pi' : X' \rightarrow X'/\mathbb{G}_m$  and  $p_2 : X \times_{X/\mathbb{G}_m} X'/\mathbb{G}_m \rightarrow X'/\mathbb{G}_m$  are isomorphic as  $\mathbb{G}_m$ -bundles over  $X'/\mathbb{G}_m$ . Since the diagram

$$\begin{array}{ccc} X & \longrightarrow & X/\mathbb{G}_m \\ \rho \downarrow & & \downarrow \bar{\rho} \\ \mathbb{A}_*^2 & \longrightarrow & \mathbb{P}^1 = \mathbb{A}_*^2/\mathbb{G}_m, \end{array}$$

is Cartesian, the isomorphism class of the  $\mathbb{G}_m$ -bundle  $X \rightarrow X/\mathbb{G}_m$  in  $H^1(X/\mathbb{G}_m, \mathbb{G}_m) \simeq \text{Pic}(X/\mathbb{G}_m)$  coincides with  $\bar{\rho}^* \mathcal{O}_{\mathbb{P}^1}(-1)$ , and similarly for  $X' \rightarrow X'/\mathbb{G}_m$ . So the  $\mathbb{G}_m$ -equivariant isomorphism of  $X$  and  $X'$  reduces to the existence of an isomorphism  $f : X'/\mathbb{G}_m \xrightarrow{\sim} X/\mathbb{G}_m$  such that  $f^* \bar{\rho}^* \mathcal{O}_{\mathbb{P}^1}(-1) = \bar{\rho}'^* \mathcal{O}_{\mathbb{P}^1}(-1)$ . Since  $\bar{\rho} : X/\mathbb{G}_m \rightarrow \mathbb{P}^1$  and  $\bar{\rho}' : X'/\mathbb{G}_m \rightarrow \mathbb{P}^1$  are both nontrivial  $\mathcal{O}_{\mathbb{P}^1}(-d)$ -bundles, the Danilov-Gizatullin Theorem [10, Theorem 5.8.1 and Remark 4.8.6] implies directly the existence of an isomorphism  $f : X'/\mathbb{G}_m \xrightarrow{\sim} X/\mathbb{G}_m$ . Moreover, since  $f^*$  maps generators of  $\text{Pic}(X/\mathbb{G}_m)$  to generators of  $\text{Pic}(X'/\mathbb{G}_m)$ , it follows that  $X'$  and  $X \times_{X/\mathbb{G}_m} X'/\mathbb{G}_m$  are isomorphic as locally trivial  $\mathbb{A}_*^1$ -bundles over  $X'/\mathbb{G}_m$ , which implies the second assertion of the theorem. Finally, the existence of an isomorphism  $f$  with the required additional property is guaranteed by Theorem 3.1 in the Appendix.  $\square$

**Example 2.4.** As a consequence of Theorem 2.3 above, we get in particular that if  $m + n = m' + n'$ , then the varieties  $X_{m,n}$  and  $X_{m',n'}$  are isomorphic. While it appears to be rather difficult to construct an explicit isomorphism between even the first interesting examples  $X_{2,2}$  and  $X_{3,1}$ , one can check that the morphism

$$\pi : X_{2,2} = \{x^2v - y^2u = 1\} \rightarrow \mathbb{A}_*^2 = \text{Spec}(\mathbb{C}[a, b] \setminus \{(0, 0)\}), (x, y, u, v) \mapsto (x - \frac{1}{2}y, \frac{6x - y}{8}v - \frac{3y - 2x}{2}u)$$

is  $\mathbb{A}^1$ -bundle isomorphic to the one  $\rho : X_{3,1} \simeq \{a^3v' - bu' = 1\} \rightarrow \mathbb{A}_*^2$ . More precisely, letting

$$w = \frac{5}{16}v^2x + u^2x + \frac{5}{2}vux - \frac{1}{32}v^2y - \frac{5}{2}u^2y - \frac{5}{4}vuy \in \Gamma(X_{2,2}, \mathcal{O}_{X_{2,2}}),$$

a direct computation shows that  $\pi^{-1}(U_a) \simeq U_a \times \text{Spec}(\mathbb{C}[a^{-3}(y + a + ab)])$ ,  $\pi^{-1}(U_b) \simeq U_b \times \text{Spec}(\mathbb{C}[b^{-1}w])$  and that  $a^{-3}(y + a + ab) - b^{-1}w = a^{-3}b^{-1} \in \Gamma(\pi^{-1}(U_a \cap U_b), \mathcal{O}_{X_{2,2}})$ . Thus  $\pi : X_{2,2} \rightarrow \mathbb{A}_*^2$  is an  $\mathbb{A}^1$ -bundle with the same associated Čech cocycle  $a^{-3}b^{-1} \in C^1(\{U_a, U_b\}, \mathcal{O}_{\mathbb{A}_*^2})$  as  $\rho : X_{3,1} \rightarrow \mathbb{A}_*^2$ .

### 2.3. Existence of exotic affine spheres.

Here we show that exotic affine spheres occur among the total spaces of non trivial  $\mathbb{A}^1$ -bundles over  $\mathbb{A}_*^2$ .

**2.4.** To illustrate the idea behind the proof of Theorem 2.5 below, let us first consider the varieties  $X_{m,1} = \{x^mv - yu = 1\}$ ,  $m \geq 1$ . Because  $X_{m,1}$  is smooth, the canonical sheaf  $\omega_m = \omega_{X_{m,1}}$  of  $X_{m,1}$  is a free  $\mathcal{O}_{X_{m,1}}$ -module generated for instance by the global nowhere vanishing 3-form

$$\alpha_m = x^{-m}dx \wedge dy \wedge du|_{X_{m,1}} = -y^{-1}dx \wedge dy \wedge dv|_{X_{m,1}}.$$

The pull back of  $\alpha_1$  by an isomorphism  $\varphi : X_{m,1} \xrightarrow{\sim} X_{1,1}$  would be a nowhere vanishing algebraic 3-form on  $X_{m,1}$  whence a nonzero scalar multiple of  $\alpha_m$  since nonzero constants are the only invertible functions  $X_{m,1}$ . On the other hand, since  $X_{m,1}$  has the real sphere  $S^3$  as a strong deformation retract, the de Rham cohomology group  $H_{dR}^3(X_{m,1}, \mathbb{C})$  is one dimensional over  $\mathbb{C}$ . Using the fact that the de Rham cohomology of a smooth complex affine variety equals the cohomology of its algebraic de Rham complex [12], it can be checked directly that  $H_{dR}^3(X_{m,1}, \mathbb{C}) \simeq \Omega_{X_{m,1}}^3/d\Omega_{X_{m,1}}^2$  is spanned by the class of  $x^{m-1}\alpha_m = x^{-1}dx \wedge dy \wedge du|_{X_{m,1}}$ . The isomorphism  $\varphi$  would induce an isomorphism in cohomology and since  $H_{dR}^3(X_{1,1}, \mathbb{C})$  is spanned by the class of  $\alpha_1$  it would follow that  $H_{dR}^3(X_{m,1}, \mathbb{C})$  is spanned by the class of  $\alpha_m$  too. This is absurd since for every  $m \geq 2$ ,  $\alpha_m$  is an exact form, having for instance the global 2-form

$$\frac{dy \wedge du}{(1-m)x^{m-1}}|_{X_{m,1}} = \frac{xdy \wedge dv - mvd x \wedge dy}{(1-m)y}|_{X_{m,1}}$$

as a primitive.

**2.5.** More generally, recall that every non trivial  $\mathbb{A}^1$ -bundle  $\rho : X \rightarrow \mathbb{A}_*^2$  is isomorphic to one of the form

$$X(m, n, p) = \{x^mv - y^nu = p(x, y)\} \setminus \{x = y = 0\} \subset \mathbb{A}_*^2 \times \text{Spec}(\mathbb{C}[u, v])$$

where  $p(x, y) \in \mathbb{C}[x, y]$  is a polynomial divisible neither by  $x$  nor by  $y$  and satisfying  $\deg_x p < m$  and  $\deg_y p < n$ . It turns out that varieties  $X(m, n, p)$  corresponding to a polynomial  $p$  of maximum possible degree  $m + n - 2$  form a distinguished class. Namely, we have the following result:

**Theorem 2.5.** *Let  $X_1 = X(m_1, n_1, p_1)$  and  $X_2 = X(m_2, n_2, p_2)$  be  $\mathbb{A}^1$ -bundles as above. If  $\deg p_1 = m_1 + n_1 - 2$  but  $\deg p_2 < m_2 + n_2 - 2$  then  $X_1$  and  $X_2$  are not isomorphic as algebraic varieties.*



*Proof.* We will show that the cohomology class in  $H_{dR}^3(X, \mathbb{C}) \simeq \mathbb{C}$  of an arbitrary nowhere vanishing algebraic 3-form  $\omega$  on  $X = X(m, n, p)$  is trivial if  $\deg p < m + n - 2$  and a generator otherwise. This prevents in particular the existence of an isomorphism  $f : X_2 \xrightarrow{\sim} X_1$ : indeed, otherwise, similarly as in the particular case above, the pull-back of a nowhere vanishing algebraic 3-form  $\omega$  on  $X_1$  would be a nowhere vanishing algebraic 3-form  $f^*\omega$  on  $X_2$  whose cohomology class  $[f^*\omega] = f^*[\omega] \in H_{dR}^3(X_2, \mathbb{C}) \simeq H_{dR}^3(X_1, \mathbb{C})$  would generate  $H_{dR}^3(X_2, \mathbb{C})$ , a contradiction. Recall 1.3 that  $X = X(m, n, p)$  is covered by two principal affine open subsets  $X_x \simeq U_x \times \text{Spec}(\mathbb{C}[t_x])$  and  $X_y \simeq U_y \times \text{Spec}(\mathbb{C}[t_y])$ , where  $t_x = x^{-m}u$  and  $t_y = y^{-n}v$ . Since every invertible function on  $X$  is constant, a nowhere vanishing algebraic 3-form  $\omega$  on  $X$  is uniquely determined locally by a pair of 3-forms  $\omega|_{X_x} = \lambda dx \wedge dy \wedge dt_x \in \Omega_{U_x \times \mathbb{A}^1}^3$  and  $\omega|_{X_y} = \lambda dy \wedge dx \wedge dt_y \in \Omega_{U_y \times \mathbb{A}^1}^3$ , where  $\lambda \in \mathbb{C}^*$ . Let  $(\alpha_x, \alpha_y) = (\lambda t_x dx \wedge dy, \lambda t_y dx \wedge dy) \in \Omega_{U_x \times \mathbb{A}^1}^2 \times \Omega_{U_y \times \mathbb{A}^1}^2$  be local primitives of  $\omega|_{X_x}$  and  $\omega|_{X_y}$  respectively. By definition of the connecting homomorphism

$$\delta : H_{dR}^2(X_x \cap X_y, \mathbb{C}) \simeq H_{dR}^2(U_x \cap U_y, \mathbb{C}) \xrightarrow{\sim} H_{dR}^3(X, \mathbb{C})$$

in the Mayer-Vietoris long exact sequence for the covering of  $X$  by  $X_x$  and  $X_y$ , the cohomology class of  $\omega \in \Omega_X^3$  in  $H_{dR}^3(X, \mathbb{C}) \simeq \Omega_X^3/d\Omega_X^2$  coincides with the image by  $\delta$  of the cohomology class  $\alpha \in H_{dR}^2(X_x \cap X_y, \mathbb{C})$  of the 2-form

$$\begin{aligned} (\alpha_y - \alpha_x)|_{X_x \cap X_y} &= (\lambda t_y dx \wedge dy - \lambda t_x dx \wedge dy)|_{X_x \cap X_y} \\ &= \lambda(t_x + x^{-m}y^{-n}p(x, y))dx \wedge dy - \lambda t_x dx \wedge dy \\ &= \lambda x^{-m}y^{-n}p(x, y) dx \wedge dy \in \Omega_{X_x \cap X_y/\mathbb{C}}^2. \end{aligned}$$

Such a form is exact if and only if  $x^{-m}y^{-n}p(x, y)$  does not contain a term of the form  $ax^{-1}y^{-1}$ , where  $a \in \mathbb{C}^*$ , that is, if and only if  $\deg p < m + n - 2$ . Thus  $[\omega] = \delta(\alpha) \in H_{dR}^3(X, \mathbb{C})$  is either trivial if  $\deg p < m + n - 2$  or a generator otherwise.  $\square$

**Corollary 2.6.** *The total space of a nontrivial homogeneous  $\mathbb{G}_a$ -bundle  $\rho : X \rightarrow \mathbb{A}_*^2$  of degree  $-d < -2$  is not isomorphic to  $X_{1,1} \simeq \text{SL}_2(\mathbb{C})$ .*

*Proof.* The variety  $X_{1,1} = X(1, 1, 1) = \{xv - yu = 1\}$  belongs to the class  $X(m, n, p)$  with  $\deg p = m + n - 2$ . On the other hand, by virtue of Theorem 2.3, we may assume that  $X \simeq X(d-1, 1, 1) = \{x^{d-1}v - yu = 1\}$ . Since  $0 = \deg p < d - 2$  by hypothesis, the assertion follows from Theorem 2.5 above.  $\square$

**Example 2.7.** Theorem 2.5 implies that the total spaces of the  $\mathbb{G}_a$ -bundles

$$X = \{x^2v - y^2u = 1\} \rightarrow \mathbb{A}_*^2 \quad \text{and} \quad X' = \{x^2v - y^2u = 1 + xy\} \rightarrow \mathbb{A}_*^2$$

are not isomorphic as algebraic varieties. However,  $X$  and  $X'$  are biholomorphic  $\mathbb{A}^1$ -bundles over  $\mathbb{A}_*^2$ . Indeed, the Čech 1-cocycle  $(xy)^{-2}(1 + xy) \in C^1(\{U_x, U_y\}, \mathcal{H}ol_{\mathbb{A}_*^2})$  defining  $X'$  is analytically cohomologous to the one  $(xy)^{-2} \exp(xy)$ , obtained by multiplying the Čech 1-cocycle  $(xy)^{-2}$  defining  $X$  by the nowhere vanishing holomorphic function  $\exp(xy)$  on  $\mathbb{A}_*^2$ . In contrast with the algebraic situation considered in the proof of Theorem 2.5, the pull back by the corresponding biholomorphism  $X \xrightarrow{\sim} X'$  of the nowhere vanishing algebraic 3-form  $\omega = x^{-2}dx \wedge dy \wedge du|_{X'}$  on  $X'$ , whose class generates  $H_{dR}^3(X', \mathbb{C})$ , is the nowhere vanishing holomorphic 3-form  $x^{-2} \exp(-xy) dx \wedge dy \wedge du|_X$ . The latter is analytically cohomologous to the algebraic 3-form  $-x^{-1}y dx \wedge dy \wedge du|_X$ , whose class generates  $H_{dR}^3(X, \mathbb{C})$ .

*Remark 2.8.* Since the cylinders  $X \times \mathbb{A}^1$  and  $X' \times \mathbb{A}^1$  are algebraically isomorphic (see 1.3),  $X$  and  $X'$  above provide a new example of biholomorphic complex algebraic varieties for which algebraic cancellation fails. Of course,  $X$  and  $X'$  are remote from affine spaces from a topological point of view. But, in contrast with other families of 3-dimensional counter-examples constructed so far [6] [7],  $X$  and  $X'$  have a trivial Makar-Limanov invariant (see 1.4 above). It is interesting to relate the existing counter-examples to Miyanishi's characterization of the affine 3-space  $\mathbb{A}^3$  [21], which can be equivalently formulated as the fact that a smooth affine threefold  $X$  is algebraically isomorphic to  $\mathbb{A}^3$  if and only if it satisfies the following conditions :

- (i) There exists a regular function  $f : X \rightarrow \mathbb{A}^1$  and a Zariski open subset  $U \subset \mathbb{A}^1$  such that  $f^{-1}(U) \simeq U \times \mathbb{A}^2$ ,
- (ii) All scheme theoretic fibers of  $f$  are UFDs (i.e.  $\Gamma(X_c, \mathcal{O}_{X_c})$  is a UFD for every fiber  $X_c = f^{-1}(c)$ ,  $c \in \mathbb{A}^1$ ),
- (iii)  $H^3(X, \mathbb{Z}) = 0$ .

The counter-examples obtained in [6] for contractible affine threefolds satisfy (i) and (iii) but not (ii). On the other hand,  $X$  and  $X'$  above satisfy (i) and (ii) (by choosing for instance the projection  $\text{pr}_x$  for  $f$ ) but not (iii). The Cancellation Problem for  $\mathbb{A}^3$  itself is still open but we see that cancellation fails whenever one of the necessary conditions (ii) or (iii) above is relaxed.

## 3. APPENDIX

## 3.1. The Danilov-Gizatullin Isomorphism Theorem.

This subsection is devoted to the proof of the following result, which is a slight refinement of the so-called Danilov-Gizatullin Isomorphism Theorem.

**Theorem 3.1.** *Let  $\nu : Y \rightarrow \mathbb{P}^1$  and  $\nu' : Y' \rightarrow \mathbb{P}^1$  be non trivial principal  $\mathcal{O}_{\mathbb{P}^1}(-d)$ -bundles for a certain  $d \geq 2$ . Then there exists an isomorphism  $f : Y' \xrightarrow{\sim} Y$  such that  $f^*(\nu^*\mathcal{O}_{\mathbb{P}^1}(1)) \simeq (\nu')^*\mathcal{O}_{\mathbb{P}^1}(1)$ . In particular, for a fixed  $d \geq 2$ , the total spaces of non trivial  $\mathcal{O}_{\mathbb{P}^1}(-d)$ -bundles are all isomorphic as abstract algebraic varieties.*

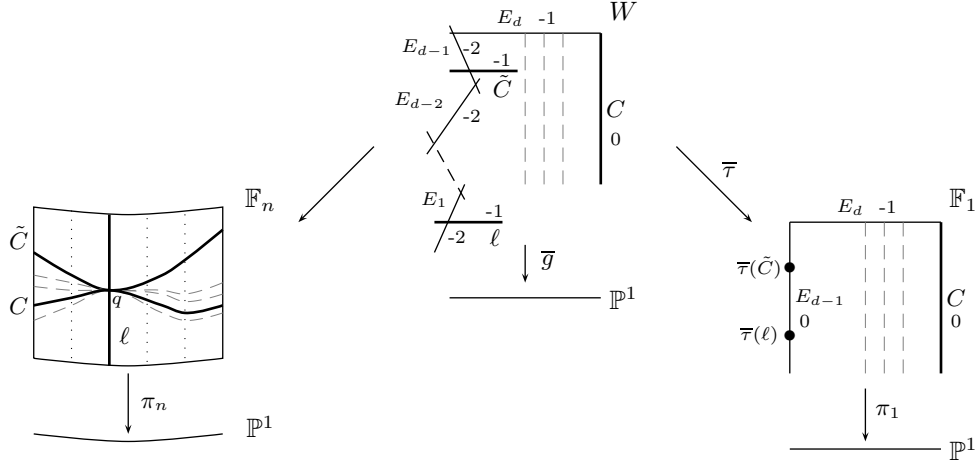
**3.1.** The Danilov-Gizatullin Theorem [10, Theorem 5.8.1] is actually stated as the fact that the isomorphism type of the complement of an ample section  $C$  in a Hirzebruch surface  $\pi_n : \mathbb{F}_n = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-n)) \rightarrow \mathbb{P}^1$ ,  $n \geq 0$ , (see e.g. [15, V.2]) depends only on the self-intersection  $C^2$  of  $C$ , whence, in particular, depends neither on the ambient surface nor on the choice of a particular section. The relation with Theorem 3.1 above is given by the observation that a non trivial  $\mathcal{O}_{\mathbb{P}^1}(-d)$ -bundle  $\nu : Y \rightarrow \mathbb{P}^1$  always arises as the complement of an ample section  $C$  with self-intersection  $C^2 = d$  in a suitable Hirzebruch surface [10, Remark 4.8.6]. Indeed, letting  $0 \rightarrow \mathcal{O}_{\mathbb{P}^1} \rightarrow \mathcal{E} \rightarrow \mathcal{O}_{\mathbb{P}^1}(d) \rightarrow 0$  be an extension of locally free sheaves on  $\mathbb{P}^1$  representing the isomorphism class of  $Y$  in  $H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-d)) \simeq \text{Ext}_{\mathbb{P}^1}^1(\mathcal{O}_{\mathbb{P}^1}(d), \mathcal{O}_{\mathbb{P}^1})$ ,  $Y$  is isomorphic to the complement in the  $\mathbb{P}^1$ -bundle  $\pi : S = \text{Proj}_{\mathbb{P}^1}(\text{Sym}(\mathcal{E})) \rightarrow \mathbb{P}^1$  of the section  $C$  determined by the surjection  $\mathcal{E} \rightarrow \mathcal{O}_{\mathbb{P}^1}(d)$ . Since  $\mathcal{E}$  is a decomposable locally free sheaf of rank 2, degree  $-d$ , equipped with a surjection onto  $\mathcal{O}_{\mathbb{P}^1}(d)$ , it is isomorphic  $\mathcal{O}_{\mathbb{P}^1}(a) \oplus \mathcal{O}_{\mathbb{P}^1}(d-a)$  for a certain  $a \in \mathbb{Z}$  such that, up to replacing  $a$  by  $d-a$ , we have either  $a = 0$  or  $d-a \geq a > 0$ . Therefore,  $S \simeq \text{Proj}_{\mathbb{P}^1}(\text{Sym}(\mathcal{E} \otimes \mathcal{O}_{\mathbb{P}^1}(a-d))) \simeq \mathbb{F}_n$ , where  $n = d-2a \geq 0$ , with the section  $C$  determined by a surjection  $\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-n) \rightarrow \mathcal{O}_{\mathbb{P}^1}(a)$ . Letting  $C_0$  be a section with self-intersection  $-n \leq 0$  and  $\ell$  be a fiber of  $\pi_n$ , we have  $C \sim C_0 + (d-a)\ell$ , which implies that  $C^2 = d$ . Furthermore,  $a = 0$  if and only if the above extension splits, that is, if and only if  $\nu : Y \rightarrow \mathbb{P}^1$  is the trivial  $\mathcal{O}_{\mathbb{P}^1}(-d)$ -bundle. Otherwise,  $d-a > n$ , and so,  $C$  is the support of an ample divisor on  $S$  [15, 2.20 p. 382].

**3.2.** The existence of an isomorphism  $f$  with the required property can actually be derived from a careful reading of the recent proof of the Danilov-Gizatullin Theorem given in [8]. However, for the convenience of the reader, we provide a complete argument. Our strategy is very similar to the one in *loc. cit.*: we establish that the total space of a non trivial  $\mathcal{O}_{\mathbb{P}^1}(-d)$ -bundle  $\nu : Y \rightarrow \mathbb{P}^1$  is equipped with a certain type of smooth fibration  $\theta : Y \rightarrow \mathbb{A}^1$  with general fibers isomorphic to  $\mathbb{A}^1$  which, for a fixed  $d \geq 2$ , admits a unique model  $\theta_d : S(d) \rightarrow \mathbb{A}^1$  up to isomorphism of fibrations. Since  $\text{Pic}(Y) \simeq \text{CaCl}(Y)$  is generated by the class of a fiber of  $\nu$ , Theorem 3.1 will then follow from the additional observation that one can always choose a special isomorphism of fibrations  $\psi : Y \xrightarrow{\sim} S(d)$  which maps a suitable fiber of  $\nu$  onto a fixed irreducible component  $\Delta$  of a fiber of  $\theta_d$ .

**3.3.** The fibration  $\theta : Y \rightarrow \mathbb{A}^1$  is constructed as follows. We may suppose that the non trivial  $\mathcal{O}_{\mathbb{P}^1}(-d)$ -bundle  $\nu : Y \rightarrow \mathbb{P}^1$  is embedded in a Hirzebruch surface  $\pi_n : \mathbb{F}_n \rightarrow \mathbb{P}^1$  for a certain  $n \geq 0$  as the complement of an ample section  $C$  with  $C^2 = d \geq 2$ . Now suppose that there exists another section  $\tilde{C}$  of  $\pi_n$  intersecting  $C$  in a unique point  $q$ , with multiplicity  $C \cdot \tilde{C} = d-1$ . Letting  $\ell = \pi_n^{-1}(\pi_n(q))$ , the divisors  $C$  and  $\tilde{C} + \ell$  are linearly equivalent and define a pencil of rational curves  $g : \mathbb{F}_n \dashrightarrow \mathbb{P}^1$  with  $q$  as a unique proper base point. This pencil restricts on  $Y = \mathbb{F}_n \setminus C$  to a smooth surjective morphism  $\theta : Y \rightarrow B = \mathbb{P}^1 \setminus \{\bar{g}(C)\} \simeq \mathbb{A}^1$  with general fibers isomorphic to  $\mathbb{A}^1$  and with a unique degenerate fiber, say  $\theta^{-1}(0)$  up to the choice of a suitable coordinate  $x$  on  $B \simeq \mathbb{A}^1$ , consisting of the disjoint union of  $\tilde{C}_0 = \tilde{C} \setminus \{q\} \simeq \mathbb{A}^1$  and  $\ell_0 = \ell \setminus \{q\} \simeq \mathbb{A}^1$ . A minimal resolution  $\bar{g} : W \rightarrow \mathbb{P}^1$  of  $g : \mathbb{F}_n \dashrightarrow \mathbb{P}^1$  is obtained from  $\mathbb{F}_n$  by blowing up  $d$  times the point  $q$ , with successive exceptional divisors  $E_1, \dots, E_d$ , the last exceptional divisor  $E_d$  being a section of  $\bar{g}$ . The proper transform of  $C$  in  $W$  is a full fiber of  $\bar{g}$ , whereas the proper transforms of  $\tilde{C}$  and  $\ell$  are both  $-1$ -curves contained in the unique degenerate fiber  $\bar{g}^{-1}(0) = E_1 + \dots + E_{d-1} + \tilde{C} + \ell$  of  $\bar{g}$ . Since  $E_1 \cup \dots \cup E_{d-1}$  is a chain of  $(-2)$ -curves, by contracting successively  $\ell$ ,  $E_1, \dots, E_{d-2}$  and  $\tilde{C}$  we obtain a birational morphism  $\bar{\tau} : W \rightarrow \mathbb{F}_1$ . The latter restricts to a morphism  $\tau : Y \simeq W \setminus C \cup \bigcup_{i=1}^d E_i \rightarrow \mathbb{F}_1 \setminus C \cup E_d \simeq B \times \mathbb{A}^1$  of schemes over  $B$ , inducing an isomorphism  $Y \setminus \tilde{C}_0 \cup \ell_0 \xrightarrow{\sim} B \setminus \{0\} \times \mathbb{A}^1$  and contracting  $\tilde{C}_0$  and  $\ell_0$  to distinct points supported on  $\{0\} \times \mathbb{A}^1 \subset B \times \mathbb{A}^1$  (see Figure 3.1 below).

**3.4.** Up to a suitable choice of coordinate on the second factor of  $B \times \mathbb{A}^1$ , we may assume that  $\tau(\tilde{C}_0) = (0, 1)$  and  $\tau(\ell_0) = (0, 0)$ . It then follows from the construction of  $\tau$  that there exists isomorphisms  $Y \setminus \ell_0 \simeq B \times \text{Spec}(\mathbb{C}[u_1])$  and  $Y \setminus \tilde{C}_0 \simeq B \times \text{Spec}(\mathbb{C}[u_2])$  for which the restrictions of  $\tau : Y \rightarrow B \times \mathbb{A}^1$  to  $Y \setminus \ell_0$  and  $Y \setminus \tilde{C}_0$  coincide respectively with the birational morphisms

$$Y \setminus \ell_0 \rightarrow B \times \mathbb{A}^1, (x, u_1) \mapsto (x, xu_1 + 1) \quad \text{and} \quad Y \setminus \tilde{C}_0 \rightarrow B \times \mathbb{A}^1, (x, u_2) \mapsto (x, x^{d-1}u_2 + \sum_{i=1}^{d-2} a_i x^i),$$

FIGURE 3.1. Resolution of the pencil  $g : \mathbb{F}_n \dashrightarrow \mathbb{P}^1$  and the contraction  $\bar{\tau} : W \rightarrow \mathbb{F}_1$ .

where the complex numbers  $a_1, \dots, a_{d-1}$  depend on the successive centers of  $\bar{\tau} : W \rightarrow \mathbb{F}_1$ . Replacing  $u_1$  by  $v_1 = u_1 - \sum_{i=1}^{d-2} a_i x^{i-1}$  yields a new isomorphism  $Y \setminus \ell_0 \simeq B \times \text{Spec}(\mathbb{C}[v_1])$ . Letting  $v_2 = u_2$ , we can eventually identify  $\theta : Y \rightarrow B$  with the surface  $\theta_d : S(d) \rightarrow B$  obtained by gluing two copies  $S_1 = B \times \text{Spec}(\mathbb{C}[v_1])$  and  $S_2 = B \times \text{Spec}(\mathbb{C}[v_2])$  of  $B \times \mathbb{A}^1$  along  $(B \setminus \{0\}) \times \mathbb{A}^1$  by the isomorphism

$$S_1 \supset (B \setminus \{0\}) \times \mathbb{A}^1 \ni (x, v_1) \mapsto (x, x^{2-d}v_1 + x^{1-d}) \in (B \setminus \{0\}) \times \mathbb{A}^1 \subset S_2.$$

**3.5.** Summing up, starting from a section  $\tilde{C}$  of  $\pi_n$  intersecting  $C$  in a unique point  $q$  with multiplicity  $d-1$ , we constructed an isomorphism  $Y = \mathbb{F}_n \setminus C \xrightarrow{\sim} S(d)$  which maps  $\nu^{-1}(\pi_n(q)) = \ell_0$  isomorphically onto the curve  $\Delta = \{x=0\} \subset S_2$ . So Theorem 3.1 eventually follows from the next lemma (see also [10, Prop. 4.8.11]), which guarantees the existence of sections  $\tilde{C}$  with the required property.

**Lemma 3.2.** *Let  $\pi_n : \mathbb{F}_n \rightarrow \mathbb{P}^1$ ,  $n \geq 0$ , be a Hirzebruch surface and let  $C \subset \mathbb{F}_n$  be an ample section with self-intersection  $d \geq 2$ . Then given a general point  $q \in C$ , there exists a section  $\tilde{C}$  such that  $C \cdot \tilde{C} = (d-1)q$ .*

*Proof.* The existence of a section  $\tilde{C}$  such that  $C \cdot \tilde{C} = (d-1)q$  for a certain  $q \in C$  is equivalent to the existence of a rational section of the induced  $\mathbb{A}^1$ -bundle  $\nu = \pi_n|_Y : Y = \mathbb{F}_n \setminus C \rightarrow \mathbb{P}^1$  with a pole of order  $d-1$  at the point  $\pi_n(q)$ . Since  $\nu : Y \rightarrow \mathbb{P}^1 = \text{Proj}(\mathbb{C}[w_0, w_1])$  is a non trivial  $\mathcal{O}_{\mathbb{P}^1}(-d)$ -bundle, we can find local trivializations  $\tau_1 : \nu^{-1}(U_{w_1}) \xrightarrow{\sim} U_{w_1} \times \text{Spec}(\mathbb{C}[u])$  and  $\tau_0 : \nu^{-1}(U_{w_0}) \xrightarrow{\sim} U_{w_0} \times \text{Spec}(\mathbb{C}[v])$  such that, letting  $z = w_0/w_1$ , the isomorphism  $\tau_0 \circ \tau_1^{-1}|_{U_{w_1} \cap U_{w_0}}$  has the form  $(z, v) \mapsto (z^{-1}, z^d u + p(z))$  for a nonzero polynomial  $p(z) \in z\mathbb{C}[z]$  of degree  $\deg p < d$ . In these trivializations, a rational section of  $\nu$  with pole of order  $d-1$  at a point  $\lambda = \pi_n(q) \in U_{w_1} \cap U_{w_0}$  is uniquely determined by a rational function  $f_1 : U_{w_1} \dashrightarrow \mathbb{A}^1$ ,  $z \mapsto (z-\lambda)^{1-d} s(z)$  such that  $\lambda \neq 0$ ,  $s(z) \in \mathbb{C}[z]$  does not vanish at  $\lambda$ , and such that  $z^d s(z) + (z-\lambda)^{d-1} p(z) \in \mathcal{O}_{\infty}(z^{d-1})$ . Indeed, the last condition guarantees that  $z^d f_1 + p(z)$  extends to a rational function  $f_0 : U_{w_0} \dashrightarrow \mathbb{A}^1$  regular at the origin, whence that the local rational sections  $f_1$  and  $f_0$  of  $\nu$  glue to a global one  $\sigma : \mathbb{P}^1 \dashrightarrow Y$  with a unique pole at  $\lambda \in U_{w_1} \cap U_{w_0}$ , of order  $d-1$ . Writing  $(z-\lambda)^{d-1} p(z) = \alpha_\lambda(z) + z^d \beta_\lambda(z)$  where  $\alpha_\lambda(z) \in \mathbb{C}[z]$  is a non zero polynomial of degree  $\deg \alpha_\lambda \leq d-1$ , we have necessarily  $s(z) = -\beta_\lambda(z)$ , which forces in turn  $s(\lambda) = \lambda^{-d} \alpha_\lambda(\lambda)$ . Letting  $p(z) = a_1 z + \dots + a_{d-1} z^{d-1}$ , a direct computation shows that

$$s(\lambda) = \sum_{i=0}^{d-2} (-1)^{i+1} \binom{d-2}{i} a_{i+1} \lambda^i.$$

Since  $r(z) = \sum_{i=0}^{d-2} (-1)^{i+1} \binom{d-2}{i} a_{i+1} z^i$  is a nonzero polynomial, it follows that for every  $\lambda \in (U_{w_1} \cap U_{w_0}) \setminus V(r(z))$  there exists a (unique) rational section  $\sigma : \mathbb{P}^1 \dashrightarrow Y$  with pole of order  $d-1$  at  $\lambda$ .  $\square$

### 3.2. A topological characterization of the affine 2-sphere.

The fact that a smooth affine surface  $X$  diffeomorphic to  $\mathbb{S}_{\mathbb{C}}^2 = \{z_1^2 + z_2^2 + z_3^2 = 1\} \subset \mathbb{A}_{\mathbb{C}}^3$  is algebraically isomorphic to  $\mathbb{S}_{\mathbb{C}}^2$  is probably folklore. We provide a proof because of the lack of an appropriate reference.

**3.6.** Let  $\mathbb{F}_0 = \mathbb{P}^1 \times \mathbb{P}^1$  with bi-homogeneous coordinates  $([u_0 : u_1], [v_0 : v_1])$ . Via the open immersion

$$j : \mathbb{S}_{\mathbb{C}}^2 \rightarrow \mathbb{F}_0, (z_1, z_2, z_3) \mapsto ([z_1 + iz_2 : z_3 + 1], [z_1 + iz_2 : 1 - z_3])$$

we may identify  $\mathbb{S}_{\mathbb{C}}^2$  with the complement in  $\mathbb{F}_0$  of the diagonal  $\Delta = \{u_0v_1 + u_1v_0 = 0\}$ . The restriction to  $\mathbb{S}_{\mathbb{C}}^2$  of the first projection  $\mathbb{F}_0 \rightarrow \mathbb{P}^1$  is a locally trivial  $\mathbb{A}^1$ -bundle, whence a trivial  $\mathbb{R}^2$ -bundle in the euclidean topology. Thus

$$H_i(\mathbb{S}_{\mathbb{C}}^2, \mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } i = 0, 2 \\ 0 & \text{otherwise.} \end{cases}$$

Furthermore, since  $\Delta^2 = 2$ , the *fundamental group at infinity*  $\pi_1^\infty(\mathbb{S}_{\mathbb{C}}^2)$  of  $\mathbb{S}_{\mathbb{C}}^2$  (see e.g. [14, 24]) is isomorphic to  $\mathbb{Z}_2$ . It turns out that these topological invariants provide a characterization of  $\mathbb{S}_{\mathbb{C}}^2$  among all smooth affine surfaces, namely:

**Theorem 3.3.** *A smooth affine surface  $X$  with the homology type and the homotopy type at infinity of  $\mathbb{S}_{\mathbb{C}}^2$  is isomorphic to  $\mathbb{S}_{\mathbb{C}}^2$ .*

*Proof.* The finiteness of  $\pi_1^\infty(X)$  implies that  $X$  has logarithmic Kodaira dimension  $\bar{\kappa}(X) = -\infty$  [13] whence that  $X$  is affine ruled. It follows that  $X$  admits a completion into a smooth, projective, birationally ruled surface  $p : V \rightarrow C$ , where  $C$  is a smooth projective curve. One can further assume that the boundary  $D := V \setminus X$  is a connected divisor with simple normal crossings that can be written as  $D = B \cup G_0 \cup G_1 \cup \dots \cup G_s$ , where  $B$  is a section of  $p$  and the  $G_i$  are disjoint trees of smooth rational curves contained in the fibers of  $p$  [20, I.2]. In particular, the dual graph of  $D$  is a tree.

The hypotheses imply that  $H_i(X, \mathbb{Z}) = 0$  for  $i = 1, 3, 4$  and so  $H^i(X, \mathbb{Z}) = 0$  for  $i = 1, 3, 4$ , whereas  $H^2(X, \mathbb{Z}) \simeq H_2(X, \mathbb{Z}) \simeq \mathbb{Z}$  by the universal coefficient Theorem. By Poincaré-Lefschetz duality, we have  $H^i((V, D), \mathbb{Z}) \simeq H_{4-i}(X, \mathbb{Z})$  and  $H_i((V, D), \mathbb{Z}) \simeq H^{4-i}(X, \mathbb{Z})$ , and so, these groups are zero for  $i = 0, 1$  and  $3$ , and isomorphic to  $\mathbb{Z}$  for  $i = 2, 4$ . From the long exact sequences of (co)homology of pairs

$$\begin{aligned} \dots \rightarrow H_*(D) \xrightarrow{\partial_*} H_*(V) \rightarrow H_*(V, D) \rightarrow H_{*-1}(D) \rightarrow \dots \\ \dots \rightarrow H^{*-1}(D) \xrightarrow{\partial^*} H^*(V, D) \rightarrow H^*(V) \rightarrow H^*(D) \rightarrow \dots \end{aligned}$$

we get  $H_3(D, \mathbb{Z}) \simeq H_3(V, \mathbb{Z}) \simeq 0$  and so  $H^1(V, \mathbb{Z}) = 0$  by Poincaré duality. Similarly,  $H^3(D, \mathbb{Z}) \simeq H^3(V, \mathbb{Z}) = 0$  and  $H_1(V, \mathbb{Z}) = 0$ . It follows that  $H^1(D, \mathbb{Z})$  and  $H_1(D, \mathbb{Z})$  are either simultaneously 0 or isomorphic to  $\mathbb{Z}$ . In the latter case  $D$  would contain a cycle of rational curves, which is impossible from the above description of  $D$ . Thus  $H_1(D, \mathbb{Z}) = 0$  and so  $D$  is a tree of nonsingular rational curves. This implies in turn that  $H^1(V, \mathcal{O}_V) = \{0\}$  for otherwise  $D$  would be contained in a fiber of the Albanese morphism  $q : V \rightarrow \text{Alb}(V)$ , in contradiction with the fact that  $D$  is the support of an ample divisor as  $X$  is affine. Since  $\pi_1^\infty(X) \simeq \pi_1(\mathbb{S}_{\mathbb{C}}^2) \simeq \mathbb{Z}_2$  by hypothesis, Theorem 1 in [14] and its proof imply that  $D$  is a chain. Therefore, up to replacing  $V$  by another minimal completion of  $X$  obtained from  $V$  by a sequence of blow-ups and blow-downs with centers outside  $X$ , we may assume that  $D = D_0 \cup D_1 \cup \dots \cup D_s$ , where  $D_i \cdot D_j = 1$  if  $|i - j| = 1$  and 0 otherwise,  $D_0^2 = D_1^2 = 0$  and  $D_i^2 \leq -2$  for every  $i = 2, \dots, s$  (see e.g. [3, Lemma 2.7 and 2.9]). With this description, one checks easily that  $\pi_1^\infty(X) \simeq \mathbb{Z}_2$  if and only if  $s = 2$  and  $D_2^2 = -2$ . By blowing-up the point  $D_0 \cap D_1$  and contracting the proper transforms of  $D_0$ ,  $D_1$  and  $D_2$ , we reach a completion  $V_0$  of  $X$  by a smooth rational curve  $B$  with self-intersection 2. It follows from Danilov-Gizatullin's classification [10] that  $V_0 \simeq \mathbb{F}_0$  and that  $B$  is of type  $(1, 1)$ . Since the automorphism group of  $\mathbb{F}_0$  acts transitively on the set of smooth curves of type  $(1, 1)$ , we finally obtain that  $X \simeq \mathbb{F}_0 \setminus \Delta \simeq \mathbb{S}_{\mathbb{C}}^2$ . □

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